Köln, Jan 2023

The rank one property for free Frobenius extensions

Ulrich Thiel (University of Kaiserslautern)
https://ulthiel.com/math

jt with Gwyn Bellamy (University of Glasgow)

Corresponding paper: arXiv: 2301.03265

§1. Introduction

Here is a matrix:

$$\begin{pmatrix}
6 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\
0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 12 & 0 & 0 & 6 & 0 & 0 & 0 \\
6 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 24 & 0 & 0 & 12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 8 & 0 \\
0 & 0 & 3 & 0 & 0 & 12 & 0 & 0 & 6 & 0 & 0 & 0 \\
12 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{pmatrix}$$

Do you notice anything special?

permute rows and colums simultaneously ("re-labeling")

Let's rearrange the matrix by <u>blocks</u>:

$$\begin{pmatrix}
3 & 6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 12 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 6 & 12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 6 & 12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 12 & 24 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7
\end{pmatrix}$$

Do you notice anything special?

permute rows and colums simultaneously ("re-labeling")

Let's rearrange the matrix by <u>blocks</u>:

$$\begin{pmatrix} 3 & 6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 12 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 6 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 6 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 12 & 24 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

Do you notice anything special? $\Rightarrow$ Each block matrix is of <u>rank one</u>!

So, in each block we just need to know <u>one</u> row.

This matrix is in fact the decomposition matrix of some finite-dimensional algebra!

complex reflection group

The finite-dimensional algebra is a restricted rational Cherednik algebra $\overline{H}_c(W)$ — Details not important!

parameter

This algebra arises as follows:

$$\begin{array}{ccc} H_{0,c}(W) & & \text{the "big" (unrestricted) RCA} \\ & & \text{(an infinite-dimensional } \mathbb{C}\text{-algebra)} \\ | & & \\ Z & & \text{the center} \\ | & & \\ R & & \text{a certain subalgebra} \end{array}$$

technicality

finite extension

Then $\overline{H}_c(W) = H_{0,c}(W) / \mathfrak{m} H_{0,c}(W)$ for a certain maximal ideal $\mathfrak{m}$ of R.

This algebra has a set of standard modules $\Delta(\lambda)$, each having a simple head $L(\lambda) = \Delta(\lambda)/\operatorname{Rad}\Delta(\lambda)$.

Let $D = \left([\Delta(\lambda) : L(\mu)] \right)_{\lambda\mu}$ be the decomposition matrix.

Conjecture (T., 2012) The decomposition matrix of $\overline{\mathsf{H}}_c(W)$ is blockwise of rank one.

Proven by Bonnafé and Rouquier (2013) as a consequence of an ingenious theory.

The proof by Bonnafé–Rouquier is intricate and adapted to the special setting of RCA.

also more elementary, more representation-theoretic

We always believed there must be a more fundamental proof that works for more general settings (examples) as well.

We discovered such a proof and such a setting now.

§2 What we have proven

First, we get rid of standard modules.

It is a fact (Bellamy–T., 2018) that we have BGG reciprocity, i.e.

$$C = D^T D,$$

where

$$C = \big(\underbrace{[P(\lambda) : L(\mu)]}_{\text{projective cover of } L(\mu)}\big)_{\lambda\mu}$$

is the Cartan matrix of $\overline{H}_c(W)$.

$\Rightarrow$ D is blockwise of rank one if and only if C is blockwise of rank one.

From now on we focus on the rank one property of the Cartan matrix.

Exists for any finite-dimensional algebra.

Consider the following setup.

(0) is a prime ideal

H a prime ring

$|$

Z the center of H is Gorenstein and integrally closed

Z is noetherian and the localization Z_m has finite injective dimension (as Z-module) for any maximal ideal m of Z

$|$

R a subring which is a reglular ring

R is noetherian and the localization R_m is a regular local ring for any maximal ideal m of R

a free Frobenius extension (finite)

H is free and finite over R and there is an isomorphism

$$\phi : {}_1H_{\alpha^{-1}} \xrightarrow{\simeq} \mathrm{Hom}_R(H,R)$$

of H-bimodules for some R-algebra automorphism α of H.

${}_1H_{\alpha^{-1}}$: right action twisted by α^{-1}

There are many examples of such extensions in Lie theory!

Math. Z. 238, 733–779 (2001)
Digital Object Identifier (DOI) 10.1007/s002090100274

Mathematische Zeitschrift

The ramification of centres: Lie algebras in positive characteristic and quantised enveloping algebras

Kenneth A. Brown, Iain Gordon

Mathematics Department, University of Glasgow, Glasgow G12 8QW, UK
(e-mail: kab@maths.gla.ac.uk; ig@maths.gla.ac.uk)

Received: 24 June 1999; in final form: 30 March 2000 /
Published online: 17 May 2001 –

Abstract. Let H be a Hopf algebra over the field k which is a finite module over a central affine sub-Hopf algebra R. Examples include enveloping algebras $U(\mathfrak{g})$ of finite dimensional k-Lie algebras $\mathfrak{g}$ in positive characteristic and quantised enveloping algebras and quantised function algebras at roots of unity. The ramification behaviour of the maximal ideals of $Z(H)$ with respect to the subalgebra R is studied, and the conclusions are then applied to the cases of classical and quantised enveloping algebras. In the case of $U(\mathfrak{g})$ for $\mathfrak{g}$ semisimple a conjecture of Humphreys [28] on the block structure of $U(\mathfrak{g})$ is confirmed. In the case of $U_\epsilon(\mathfrak{g})$ for $\mathfrak{g}$ semisimple and ϵ an odd root of unity we obtain a quantum analogue of a result of Mirković and Rumynin, [35], and we fully describe the factor algebras lying over the regular sheet, [9]. The blocks of $U_\epsilon(\mathfrak{g})$ are determined, and a necessary condition (which may also be sufficient) for a baby Verma $U_\epsilon(\mathfrak{g})$-module to be simple is obtained.

1. Introduction

1.1. Throughout k will denote an algebraically closed field. In recent years common themes have become increasingly apparent in the representation theory of three important classes of k-algebras: the enveloping algebras $U(\mathfrak{g})$ of semisimple Lie algebras $\mathfrak{g}$ in positive characteristic, the quantised enveloping algebras $U_\epsilon(\mathfrak{g})$ of semisimple Lie algebras at a root of unity ϵ, and the quantised function algebras $O_\epsilon[G]$ of semisimple groups G at a root of unity ϵ, [30,14,13]. The common structure underlying these (and other related) classes is that of a triple

$$R \subseteq Z \subseteq H \qquad (1)$$

of k-algebras, where H is a Hopf algebra with centre Z, Z being an affine domain, and R is a sub-Hopf algebra of H, contained in Z, over which H (and hence Z) are finite modules. The common strategy adopted in studying the (finite dimensional) representation theory of such an algebra is to study the finite dimensional k-algebras $H/\mathfrak{m}H$, as $\mathfrak{m}$ ranges across the maximal ideal spectrum of R.

We are very grateful to J. Jantzen, L. Lebruyn, R. Marsh, A. Premet and G. Röhrle for advice, discussions and information. This research was begun while the second author was studying for a Ph.D. at the University of Glasgow, with the support of an EPSRC Research Studentship. The research of the first author was partially supported by NATO Grant CRG 960250

1.2. In this paper we continue the approach proposed and adopted in [3], [4] of looking for general results in the above setting which can then be interpreted and applied in the specific contexts mentioned above. Our starting point here is the following. Given a maximal ideal $\mathfrak{m}$ of R, how does the ramification behaviour of the maximal ideals of Z lying over $\mathfrak{m}$ interact with the representation theory of $H/\mathfrak{m}H$? And how does this ramification behaviour vary as $\mathfrak{m}$ varies through $\mathrm{Maxspec}(R)$? We discuss these questions first in the abstract setting of a triple (1) in Sect. 2, and then consider classical and quantised enveloping algebras in Sects. 3 and 4 respectively. (An analogous discussion for $O_\epsilon[G]$, where more precise results can currently be proved than in the first two classes, is given in the sequel [5] to the present paper.)

1.3. In Sect. 2, having first noted the easy fact that, *in the setting* (1), *the unramified locus of* Maxspec(Z) *is contained in the smooth locus*, we go on in Theorem 2.5 to give a characterisation of an unramified point of Maxspec(Z) under hypotheses which are satisfied in each of the three settings mentioned above. Thus, it is the main result of [4] that the smooth locus of Maxspec(Z) coincides with the Azumaya locus of H for each of the three classes listed in (1.1); see Theorem 2.6. (The *Azumaya locus* of H consists of those maximal ideals M of Z for which H/MH is simple (artinian).) Theorem 2.5 connects ramification with representation theory: it states that *when the smooth locus of Z coincides with the Azumaya locus a maximal ideal M of Z is unramified over* $\mathfrak{m} = R \cap M$ *if and only if M is an Azumaya point and H/MH is a projective $H/\mathfrak{m}H$-module*. Define a *fully Azumaya point* $\mathfrak{m}$ of R to be a maximal ideal $\mathfrak{m}$ of R such that all the maximal ideals of Z which lie over $\mathfrak{m}$ are in the Azumaya locus. Then we shall also be concerned to identify the *fully Azumaya points* $\mathfrak{m}$ of $\mathrm{Maxspec}(R)$, and to describe the corresponding factors $H/\mathfrak{m}H$.

The second theme of Sect. 2 is the problem of describing the *blocks* of $H/\mathfrak{m}H$, for a maximal ideal $\mathfrak{m}$ of R. We point out in Proposition 2.7 that a

Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 319 (2008) 1007–1034
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Cherednik, Hecke and quantum algebras as free Frobenius and Calabi–Yau extensions

K.A. Brown [a], I.G. Gordon [b,*], C.H. Stroppel [a]

[a] *Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK*
[b] *School of Mathematics and Maxwell Institute, University of Edinburgh, Edinburgh EH9 3JZ, UK*

Received 22 August 2006
Available online 26 November 2007
Communicated by J.T. Stafford

Abstract

We show how the existence of a PBW-basis and a large enough central subalgebra can be used to deduce that an algebra is Frobenius. We apply this to rational Cherednik algebras, Hecke algebras, quantised universal enveloping algebras, quantum Borels and quantised function algebras. In particular, we give a positive answer to [R. Rouquier, Representations of rational Cherednik algebras, in: Infinite-Dimensional Aspects of Representation Theory and Applications, Amer. Math. Soc., 2005, pp. 103–131] stating that the restricted rational Cherednik algebra at the value $t = 0$ is symmetric.

Keywords: Frobenius algebras; Calabi–Yau algebras; Quantum groups; Hecke algebras; Rational Cherednik algebras

1. Introduction

1.1. In this note we will consider six types of algebras:

(I) the rational Cherednik algebra $\mathrm{H}_{0,c}$ associated to the complex reflection group W;
(II) the graded (or degenerate) Hecke algebra $\mathbf{H}_{\mathrm{gr}}$ associated to a complex reflection group W;
(III) the extended affine Hecke algebra $\mathcal{H}$ associated to a finite Weyl group W;

* Corresponding author.
E-mail addresses: kab@maths.gla.ac.uk (K.A. Brown), igordon@maths.ed.ac.uk (I.G. Gordon), cs@maths.gla.ac.uk (C.H. Stroppel).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.10.026

(IV) the quantised enveloping algebra $\mathcal{U}_\epsilon(\mathfrak{g})$, at an ℓth root of unity ϵ, of a semisimple complex Lie algebra $\mathfrak{g}$;
(V) the corresponding quantum Borel $\mathcal{U}_\epsilon(\mathfrak{g})^{\geqslant 0}$;
(VI) the corresponding quantised function algebra $\mathcal{O}_\epsilon[G]$.

These algebras share two important properties: first, they have a regular central subalgebra $\mathcal{Z}$ over which they are free of finite rank, second, they—or a closely associated algebra in case (VI)—have a basis of PBW type. The purpose of this paper is to show that these two properties are the key tools for defining an associative non-degenerate $\mathcal{Z}$-bilinear form for each of these algebras, and hence for deducing Frobenius and Calabi–Yau properties for the algebras in each class.

1.2. We prove that each pair $\mathcal{Z} \subseteq R$ in the classes (I)–(VI) is a *free Frobenius extension*. The definition and basic properties are recalled in Section 2.1 and Section 2.2—in essence, one requires $\mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z}) \cong R$ as $(\mathcal{Z}\text{–}R)$-bimodules.

1.3. When an algebra R is a free Frobenius extension of a central subalgebra $\mathcal{Z}$ then $\mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z})$ is in fact isomorphic to R *both* as a left *and* as a right R-module, but not necessarily as a bimodule. However, there is a $\mathcal{Z}$-algebra automorphism ν of R, the *Nakayama automorphism*, such that $\mathrm{Hom}_{\mathcal{Z}}(R, \mathcal{Z}) \cong {}^1R^{\nu^{-1}}$ as R-bimodules. This automorphism is unique up to an inner automorphism. We explicitly determine the Nakayama automorphisms for each case listed above: ν is trivial (i.e. inner) in cases (I) and (IV); non-trivial in cases (II), (III) and (V) and (VI).

1.4. The results summarised in Section 1.2 have immediate consequences regarding the *Calabi–Yau property* of the algebras in classes (I)–(VI). The definition and its relevance to Serre duality are recalled in Section 2.4. In particular [8], we get natural examples of so-called Frobenius functors—that is, functors which have a biadjoint. Frobenius algebras and Frobenius extensions play an important role in many different areas (see for example [23]). They give rise to Frobenius functors which are the natural candidates for constructing interesting topological quantum field theories in dimension 2 and even 3 (see for example [37]), and also provide connections between representation theory and knot theory (for example in the spirit of [22]).

1.5. Let us assume for the moment that $\mathcal{Z} \subseteq R$ is a free Frobenius extension with Nakayama automorphism ν. If I is an ideal of $\mathcal{Z}$, then it is clear from the definitions that $\mathcal{Z}/I \subseteq R/IR$ is a free Frobenius extension with Nakayama automorphism induced by ν. This applies in particular when I is a maximal ideal $\mathfrak{m}$ of $\mathcal{Z}$; since, for R in classes (I)–(VI), every simple R-module is killed by such an ideal $\mathfrak{m}$, this is relevant to the finite-dimensional representation theory of R. Thus $R/\mathfrak{m}R$ is a Frobenius algebra, which is symmetric provided the automorphism of $R/\mathfrak{m}R$ induced by ν is inner.

1.6. To define the non-degenerate associative bilinear forms mentioned in Section 1.1, we follow in each case the approach of [12, Proposition 1.2] to the study of the inclusion $\mathcal{Z} \subseteq R$ when R is the enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional restricted Lie algebra $\mathfrak{g}$ over a field k of characteristic $p > 0$, and $\mathcal{Z}$ is the Hopf centre $k\langle x^p - x^{[p]}\colon x \in \mathfrak{g}\rangle$. In the language of the present paper, it is proved there that $\mathcal{Z} \subseteq U(\mathfrak{g})$ is a free Frobenius extension, with Nakayama

Journal of Algebra 524 (2019) 35–58

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Transfer results for Frobenius extensions

Stephane Launois, Lewis Topley *

Sibson Building, The University of Kent, Canterbury, CT2 7FS, UK

A R T I C L E I N F O

Article history:
Received 15 April 2018
Available online 17 January 2019
Communicated by Gwyn Bellamy

MSC:
16W70
16G99

Keywords:
Frobenius extensions
Deformation theory
Associative algebras

A B S T R A C T

We study Frobenius extensions which are free-filtered by a totally ordered, finitely generated abelian group, and their free-graded counterparts. First we show that the Frobenius property passes up from a free-graded extension to a free-filtered extension, then also from a free-filtered extension to the extension of their Rees algebras. Our main theorem states that, under some natural hypotheses, a free-filtered extension of algebras is Frobenius if and only if the associated graded extension is Frobenius. In the final section we apply this theorem to provide new examples and non-examples of Frobenius extensions.

1. Introduction

Throughout this paper $\Bbbk$ is a field and all algebras are $\Bbbk$-algebras. A finite dimensional algebra $\mathcal{R}$ is called a *classical Frobenius algebra* if the dual of the right regular module is isomorphic to the left regular module $(\mathcal{R}_{\mathcal{R}})^* \cong {}_{\mathcal{R}}\mathcal{R}$. Equivalently $\mathcal{R}$ admits a linear map $\mathcal{R} \to \Bbbk$ whose kernel contains no left or right ideals – we call this the *Frobenius form of* $\mathcal{R}$. The representation theory of classical Frobenius algebras admits extremely nice

* Corresponding author.
E-mail addresses: S.Launois@kent.ac.uk (S. Launois), L.Topley@kent.ac.uk (L. Topley).

https://doi.org/10.1016/j.jalgebra.2019.01.006

duality properties. For instance, it is known that the projective and injective modules coincide and, in particular, the left regular module is injective. Three notable families of examples include the group algebras of finite groups, reduced enveloping algebras of restricted Lie algebras and semidirect products $\mathcal{R} \ltimes \mathcal{R}^*$ where $\mathcal{R}$ is any Artinian ring [1, pp. 127], [8, Proposition 1.2].

A natural generalisation of a classical Frobenius algebra is the notion of a *Frobenius extension* (of the first kind), where $\Bbbk$ is replaced by some subring of $\mathcal{R}$, not necessarily central. More precisely we say that a ring extension $\mathcal{S} \subseteq \mathcal{R}$ is a Frobenius extension in case $\mathcal{R}$ is a projective left $\mathcal{S}$-module and $\mathcal{R} \cong \mathrm{Hom}_{\mathcal{S}}({}_{\mathcal{S}}\mathcal{R}, {}_{\mathcal{S}}\mathcal{S})$ as $\mathcal{R}$-$\mathcal{S}$-bimodules. Nakayama and Tsuzuku observed that Frobenius extensions can be characterised by the existence of a $\mathcal{S}$-$\mathcal{S}$-bimodule homomorphism $\Phi : \mathcal{R} \to \mathcal{S}$ generalising the Frobenius form of a classical Frobenius algebra [22], and in this paper we call this map *the Frobenius form of the extension.* Frobenius extensions play an important role in a diverse array of topics, such as link invariants and 2-dimensional TQFT, as well as having many applications in the representation theory of Hopf algebras (see [13] for a survey). The examples which we will be interested in are quantum groups which are free of finite type over their centre, as well as some important new families arising in modular representation theory. For this reason we focus on Frobenius extensions $\mathcal{S} \subseteq \mathcal{R}$ where $\mathcal{S}$ is a subalgebra of the centre of $\mathcal{R}$. It seems plausible that some of our results could be extended to weaken this hypothesis.

Brown–Gordon–Stroppel gave many new examples of Frobenius extensions [3]. Their approach was fairly uniform: in each case they gave an example of a Frobenius form $\Phi : \mathcal{R} \to \mathcal{S}$ and checked the defining property via a single simple hypothesis. In [3, 1.6] they asked whether there exists an axiomatic approach which would apply to all of their examples simultaneously, and it was this question which provided the first motivation for our work. One feature shared by many of their examples, as well as other classical examples, is a filtration by a totally ordered finitely generated abelian group G, and in this paper we develop general tools which might help to prove the Frobenius property in the presence of such a filtration. For the rest of the introduction we fix such a group G and we use the words *graded* and *filtered* to mean G-graded and G-filtered.

When dealing with filtrations and gradings it is natural to require that the module structures carry filtrations which are compatible with the actions: such modules are known as *free-filtered* and *free-graded* modules respectively (see §2.4 for an introduction). *Free filtered* and *free-graded* ring extensions are defined in the obvious manner. When $\mathcal{R}$ is a filtered algebra we write $\mathrm{gr}\,\mathcal{R} := \bigoplus_{g\in G}(\mathcal{R}_g / \sum_{g>h\in G} \mathcal{R}_h)$ for the associated graded algebra.

Now suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a central extension. We say that $\mathcal{S} \subseteq \mathcal{R}$ is a *free-graded Frobenius extension* if $\mathcal{S} \subseteq \mathcal{R}$ is a free-graded extension equipped with a homogeneous Frobenius form $\Phi : \mathcal{R} \to \mathcal{S}$. Similarly we say that $\mathcal{S} \subseteq \mathcal{R}$ is a *free-filtered Frobenius extension* if $\mathcal{S} \subseteq \mathcal{R}$ is a free-filtered ring extension and a Frobenius extension.

at least one of the terms of highest (total) degree is monic

Since H is finite over its center Z, it is a <u>PI ring</u>, i.e. there is a monic (non-commutative) polynomial $f \in Z\langle x_1, \dots, x_m\rangle$ for some m such that $f(h_1, \dots, h_m) = 0$ for all $h_1, \dots, h_m \in H$.

The <u>PI-degree</u> of H is $d := \frac{1}{2} \cdot \underbrace{\text{minimal degree}}_{}$ of such polynomials (using Posner's theorem).

will be even (Kaplansky's theorem)

We assume that $d \cdot 1_H$ is <u>invertible</u> in H. (This holds for example if H is an algebra over a field of characteristic not dividing d)

Now, let $\mathfrak{m}$ be a maximal ideal of R.

Then $A := H/\mathfrak{m}H$ is a finite-dimensional algebra over the field $K := R/\mathfrak{m}$.

all simple modules remain simple under field extensions

Assume that A <u>splits</u> over K. (This holds for example if K is algebraically closed. And this holds for example for any $\mathfrak{m}$ if R is a finite type algebra over an algebraically closed field).

Let C be the Cartan matrix of A. Let $C_1, \dots, C_k$ be the blocks of C.

Let p be the characteristic of $K = R/\mathfrak{m}$.

integral matrix

By the <u>p-rank</u> of C_i we mean the rank of the reduction of C_i mod p.

Note that for $p=0$ this is the usual rank.

We assume that the p-rank of C_i is <u>non-zero</u> for all i. This holds for example if $p=0$ (since C_i is never the zero matrix, hence rank ≥ 1).

<u>Theorem (Bellamy–T., 2023)</u> Each C_i is of p-rank one.

<u>Example:</u> The restricted RCA $\overline{H}_c(W)$ fits into our setting

<u>Another example:</u> Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

Let $U_\xi(\mathfrak{g})$ be the (simply connected) <u>quantized enveloping algebra</u> of $\mathfrak{g}$ at an ℓ-th root of unity $\xi \in \mathbb{C}$.
(We assume that $\ell \geq 3$ is an odd integer, prime to 3 if $\mathfrak{g}$ contains a factor of type G_2.)

Let Z be the center and let R be the <u>ℓ-center</u> (a certain central Hopf subalgebra).

Then $U_\xi(\mathfrak{g}) \supset Z \supset R$ fits into our setting. One particular quotient is the <u>restricted quantized enveloping algebra</u> $\overline{U}_\xi(\mathfrak{g})$

<u>More examples</u> in our paper!

Remark on positive characteristic

Let G be a connected reductive group over an algebraically closed field of characteristic $p>0$.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $H = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$.

The PI degree d of H is p^N, where $2N = |\underbrace{\Phi}_{\text{root system of } G}|$

$\Rightarrow d \cdot 1_H$ is not invertible $\Rightarrow$ does not fit into our setting!

More on this in our paper.

§3 Some ideas about the proof

A key ingredient in our proof is from:

JOURNAL OF ALGEBRA 188, 476–505 (1997)
ARTICLE NO. JA966827

Representations of Finite-Dimensional Hopf Algebras

Martin Lorenz*

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122-6094

Communicated by Susan Montgomery

Received April 3, 1995

Let H denote a finite-dimensional Hopf algebra with antipode S over a field $\Bbbk$. We give a new proof of the fact, due to Oberst and Schneider [*Manuscripta Math.* 8 (1973), 217–241], that H is a symmetric algebra if and only if H is unimodular and S^2 is inner. If H is involutory and not semisimple, then the dimensions of all projective H-modules are shown to be divisible by char $\Bbbk$. In the case where $\Bbbk$ is a splitting field for H, we give a formula for the rank of the Cartan matrix of H, reduced mod char $\Bbbk$, in terms of an integral for H. Explicit computations of the Cartan matrix, the ring structure of $G_0(H)$, and the structure of the principal indecomposable modules are carried out for certain specific Hopf algebras, in particular for the restricted enveloping algebras of completely solvable p-Lie algebras and of $sl(2, \Bbbk)$.

Communications in Algebra®, 39: 4733–4750, 2011

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2011.617619

Taylor & Francis
Taylor & Francis Group

PROJECTIVE MODULES OVER FROBENIUS ALGEBRAS AND HOPF COMODULE ALGEBRAS

Martin Lorenz[1] and Loretta Fitzgerald Tokoly[2]

[1]*Department of Mathematics, Temple University, Philadelphia, Pennsylvania, USA*
[2]*Mathematics Division, Howard Community College, Columbia, Maryland, USA*

This note presents some results on projective modules and the Grothendieck groups K_0 and G_0 for Frobenius algebras and for certain Hopf Galois extensions. Our principal technical tools are the Higman trace for Frobenius algebras and a product formula for Hattori-Stallings ranks of projectives over Hopf Galois extensions.

They have shown: if A is a (finite-dimensional) Frobenius algebra over a field K and A splits over K then

$$p\text{-rank of the Cartan matrix } C \text{ of } A = \operatorname{rank} \tau'$$

Here, $\tau' : A \to A$ is the Higman map: $\tau'(x) = \sum_i g_i' x h_i'$, where $\{g_i'\}$ and $\{h_i'\}$ is a pair of dual bases of A with respect to the Frobenius structure.

projective class group

Grothendieck group

$$\begin{tikzcd}
K_0(A) \otimes_{\mathbb{Z}} K \arrow[r, "C \bmod p"] \arrow[d, "\simeq", "\text{Hattori-Stallings rank}"'] & G_0(A) \otimes_{\mathbb{Z}} K \arrow[d, "\text{character}", "\simeq"'] \\
A/[A,A] \arrow[r] & (A/[A,A])^*
\end{tikzcd}$$

vertical maps are isos because A splits over K

$\mathbb{Z}$-submodule generated by commutators

$$a \longmapsto (b \longmapsto \operatorname{Tr}_{A/K}(L_b \circ R_a))$$

left/right multiplication

rank of this map = rank of τ'

Problem: We do not know rank τ' a priori.

What we have proven: if A arises from our (global) setting then

$$\operatorname{rank} \tau' = \text{number of blocks of } A$$

So,

$$p\text{-rank of } C = \text{number of blocks of } A$$

Since we assume that the p-rank of each bloch matrix C_i is non-zero, this implies that each C_i is of p-rank one.

Here are some ideas how we proved that $\operatorname{rank} \tau' =$ number of blocks of A.

Since $R \subset H$ is a Frobenius, we also have a Higman map $\tau : H \to H$.

The automorphism $\alpha : H \to H$ induces an automorphism $\alpha' : A \to A$ making A a Frobenius algebra over $\overset{K}{\overset{\|}{R/m}}$.

If $\tau' : A \to A$ denotes the Higman map then the diagram

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "\tau"'] & A \arrow[d, "\tau'"] \\
H \arrow[r] & A
\end{tikzcd}$$

commutes.

It is a general fact that the image of τ is contained in the <u>Nakayama center</u>

$$Z_\alpha = \{ h \in H \mid ha = \alpha(a) h \text{ for all } a \in H \}$$

of H. This is a Z-module.

Higman center of A

Similarly, the image of τ' is contained in $Z_{\alpha'}$. It is then clear that

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\tau"'] & A \arrow[d, "\tau'"] \\ Z_\alpha \arrow[r] & Z_{\alpha'} \end{tikzcd}$$

commutes.

Let Z', resp. Z_α', be the image of Z, resp. Z_α, in $A = H/_{\mathfrak{m}H}$.

Clearly, Z' is contained in the center of A but it may not be the whole center.

Nonetheless, Z' contains all the block idempotents of A by Müller's theorem (1976).

We have shown that:

uses H prime, Z integrally closed, $d \cdot 1_H$ invertible + a result by Braun (2007)

1) Z and Z_α are direct summands of H as Z-modules.

$\Rightarrow$ the natural maps $Z/_{\mathfrak{m}Z} \longrightarrow Z'$ and $Z_\alpha/_{\mathfrak{m}Z_\alpha} \longrightarrow Z_\alpha'$ are isomorphisms

uses that $R_{\mathfrak{m}}$ is regular

2) Z' is Frobenius (over $K = R/_{\mathfrak{m}}$)

uses $Z_{\mathfrak{m}}$ Gorenstein

3) Z_α' is a free Z'-module of rank one

4) $\operatorname{Im} \tau' \subset \operatorname{Soc}_{Z'} Z_\alpha'$

Since A splits over K, so does Z' $\Rightarrow$ every simple Z'-module is one-dimensional over K.

Hence,

$$\dim_K \operatorname{Im}\tau' \overset{4)}{\leq} \dim_K \operatorname{Soc}_{Z'} Z'_\alpha = \overset{\text{length}}{\ell}(\operatorname{Soc}_{Z'} Z'_\alpha) \overset{3)}{=} \ell(\operatorname{Soc}_{Z'} Z') \overset{2)}{=} |\overset{\text{set of blocks}}{Bl}(Z')| \overset{\text{Müller}}{=} |Bl(A)|$$

On the other hand,

$$\dim_K \operatorname{Im}\tau' \overset{\text{Lorentz-Fitzgerald Tohdy}}{=} p\text{-rank of } C \overset{\text{p-rank of every block of C non-zero.}}{\geq} |Bl(A)|$$

So, together we have

$$\dim_K \operatorname{Im}\tau' = |Bl(A)|$$

∎