Cellularity of endomorphism algebras of tilting objects

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Jul 17, 2023. Spetses (!)

Overview

History

1. Graham, J. J. & Lehrer, G. I. (1996). Cellular algebras. *Invent. Math.*, *123*(1), 1–34.
2. Du, J. & Rui, H. (1998). Based algebras and standard bases for quasi-hereditary algebras. *Trans. Amer. Math. Soc.*, *350*(8), 3207–3235.
3. Andersen, H. H., Lehrer, G. I., & Zhang, R. (2015). Cellularity of certain quantum endomorphism algebras. *Pacific J. Math.*, *279*(1–2), 11–35.
4. Andersen, H. H., Stroppel, C., & Tubbenhauer, D. (2018). Cellular structures using U_q-tilting modules. *Pacific J. Math.*, *292*(1), 21–59.
5. Bellamy, G. & Thiel, U. (2022). Cellularity of endomorphism algebras of tilting objects. *Adv. Math.* *404*, Paper No. 108387.

The key picture

The following picture illustrates how Andersen, Stroppel, and Tubbenhauer constructed their cellular bases:

$$\begin{tikzcd}
 & \Delta(\lambda) \arrow[d, hook, "i^{\lambda}"'] \arrow[dr, "g"] & \\
T \arrow[r, "\hat{f}"] \arrow[dr, "f"'] & T(\lambda) \arrow[r, "\hat{g}"'] \arrow[d, two heads, "\pi^{\lambda}"] & T \\
 & \nabla(\lambda) &
\end{tikzcd}$$

T: a tilting module for a quantum group U_q

$T(\lambda)$: the indecomposable tilting U_q-module at λ

$\Delta(\lambda)$ and $\nabla(\lambda)$: the standard and costandard modules at λ

What we proved

We extended the AST construction from tilting modules for quantum groups to a general categorical setting.

Let $\mathcal{C}$ be a standard category (e.g. a highest weight category with finitely many simples).

Theorem

For any tilting object $T \in \mathcal{C}$ one can construct a standard basis on the algebra $\mathrm{End}_{\mathcal{C}}(T)$ as in the picture.

Theorem

If $\mathcal{C}$ is equipped with a standard duality $\mathbb{D}$, then the construction can be done in such a way that the resulting basis is cellular with respect to the anti-involution on $\mathrm{End}_{\mathcal{C}}(T)$ induced by $\mathbb{D}$. In particular, $\mathrm{End}_{\mathcal{C}}(T)$ is a cellular algebra.

Implications

Theorem

The Hecke algebra associated to a complex reflection group (à la Broué, Malle, and Rouquier) admits a natural standard basis. For a finite Coxeter group, there is a cellular basis.

In particular, this reproves (over the complex numbers only, but *not* assuming Lusztig's P1–P15): Geck, M. (2007). Hecke algebras of finite type are cellular. *Invent. Math.*, *169*(3), 501–517.

Fundamental question about the nature of cellular algebras

Is every cellular algebra the endomorphism algebra of a tilting object?

This may finally shed a categorical light on cellularity.

Cellular algebras

Standard bases (à la Du–Rui)

Throughout: E is a finite-dimensional algebra over a field K.

Definition

A standard basis of E is a K-basis $\mathcal{B}$ of E which is fibered over a poset Λ, i.e., $\mathcal{B} = \coprod_{\lambda \in \Lambda} \mathcal{B}^\lambda$, together with indexing sets $\mathcal{I}^\lambda$ and $\mathcal{J}^\lambda$ for any $\lambda \in \Lambda$ such that

$$\mathcal{B}^\lambda = \{c_{ij}^\lambda \mid (i,j) \in \mathcal{I}^\lambda \times \mathcal{J}^\lambda\} \ ,$$

and for any $\varphi \in E$ and $c_{ij}^\lambda \in \mathcal{B}^\lambda$ we have

$$\varphi \cdot c_{ij}^\lambda \equiv \sum_{k \in \mathcal{I}^\lambda} r_k^\lambda(\varphi, i) c_{kj}^\lambda \mod E^{<\lambda} \ ,$$

$$c_{ij}^\lambda \cdot \varphi \equiv \sum_{l \in \mathcal{J}^\lambda} r_l^\lambda(j, \varphi) c_{il}^\lambda \mod E^{<\lambda} \ ,$$

where $r_k^\lambda(\varphi, i), r_l^\lambda(j, \varphi) \in K$ are independent of j and i, respectively. Here, $E^{<\lambda}$ is the subspace of E spanned by the set $\bigcup_{\mu < \lambda} \mathcal{B}^\mu$.

Cellular bases (à la Graham–Lehrer)

Definition

A cellular basis of E is a standard basis $\mathcal{B}$ together with an algebra anti-involution ι on E such that $\mathcal{I}^\lambda = \mathcal{J}^\lambda$ for all $\lambda \in \Lambda$ and

$$\iota(c_{ij}^\lambda) = c_{ji}^\lambda$$

for all $(i,j) \in \mathcal{I}^\lambda \times \mathcal{J}^\lambda$.

Remarks

1. A standard/cellular basis is a structure on E.
2. While admitting a cellular basis is a restrictive property on algebras (e.g. the Cartan determinant must be positive), admitting a standard basis is not (Koenig–Xi).
3. Nonetheless, a standard basis leads to Specht modules

$$W(\lambda) := K \cdot \{a_i^\lambda \mid i \in \mathcal{I}^\lambda\}\ , \quad \varphi \cdot a_i^\lambda := \sum_{k \in \mathcal{I}^\lambda} r_k^\lambda(\varphi, i) a_k^\lambda\ ,$$

and they may involve interesting combinatorics.

Examples of cellular algebras

1. Matrix algebras: $\iota = -^t$, $\Lambda := \{\star\}$, $\mathcal{I}^\star := \mathcal{J}^\star := \{1, \dots, n\}$, $c_{ij}^\star := E_{ij}$:
$$E_{kl}E_{ij} = \delta_{li}E_{kj} \ .$$
2. Group algebra of a symmetric group, Hecke algebras of type A
3. Temperley–Lieb algebras, Brauer algebras
4. $\dots$

Observation (Andersen–Stroppel–Tubbenhauer)

All these examples arise as $\mathrm{End}_{U_q\text{-mod}}(T)$ for a tilting module T and U_q a quantum group.

Their construction usually yields different cellular bases than the usual ones though (the unit 1 is *not* a basis element).

Problems of generalizing the AST construction

The AST construction should basically work for tilting objects in any highest weight category. But there are some subtleties:

1. The proof relies on weight space decomposition of U_q-modules.
2. Tilting modules need to behave as in Ringel's theory.
3. The construction also works with U_q-mod in positive characteristic: not enough injectives, hence not highest weight.
4. Where does the involution come from? Likely from a duality on the category. But an arbitrary duality does not necessarily induce an involution on the endomorphism algebra!

Standard categories

Abelian + ordered simple objects

We require categories with standard, costandard, and tilting objects behaving in the desired way.

We came up with the concept of standard categories.

(SC1) $\mathcal{C}$ is an essentially small and locally finite abelian category over a field K.

(Locally finite: all objects are of finite length and all Hom-spaces are finite-dimensional.)

(SC2) There is a complete set $\{L(\lambda)\}_{\lambda\in\Lambda}$ of representatives of isomorphism classes of simple objects of $\mathcal{C}$ indexed by a set Λ equipped with a partial order $\leq$.

Standard and costandard objects

Definition

A costandard object for $L(\lambda)$ is an object $\nabla(\lambda)$ such that $\operatorname{Soc} \nabla(\lambda) \simeq L(\lambda)$ and all composition factors $L(\mu)$ of $\nabla(\lambda)/\operatorname{Soc} \nabla(\lambda)$ satisfy $\mu < \lambda$.

A standard object for $L(\lambda)$ is an object $\Delta(\lambda)$ such that $\operatorname{Hd} \Delta(\lambda) \simeq L(\lambda)$ and all composition factors $L(\mu)$ of $\operatorname{Rad} \Delta(\lambda)$ satisfy $\mu < \lambda$.

(SC3) Each $L(\lambda)$ has a costandard object $\nabla(\lambda)$ and a standard object $\Delta(\lambda)$ such that the following condition holds for all $\lambda, \mu \in \Lambda$ and $0 \leq i \leq 2$:

$$\operatorname{Ext}^i_{\mathcal{C}}(\Delta(\lambda), \nabla(\mu)) = \begin{cases} K & \text{if } i = 0 \text{ and } \lambda = \mu \ , \\ 0 & \text{else} \ . \end{cases}$$

(We do not need to have enough injectives for this! The Ext-groups are isomorphic to the ones in the Ind-completion of $\mathcal{C}$, and the latter behave as usual.)

Tilting objects

Definition

An object $T \in \mathcal{C}$ is tilting if it admits both a filtration whose quotients are standard objects and a filtration whose quotients are costandard objects.

The category $\mathcal{C}^t$ of tilting objects is a Krull–Schmidt category.

(SC4) For any $\lambda \in \Lambda$ there is an indecomposable object $T(\lambda) \in \mathcal{C}^t$ such that:

1. if $[T(\lambda) : L(\mu)] \neq 0$, then $\mu \leq \lambda$, and $[T(\lambda) : L(\lambda)] = 1$;
2. there is a monomorphism $\Delta(\lambda) \hookrightarrow T(\lambda)$;
3. there is an epimorphism $T(\lambda) \twoheadrightarrow \nabla(\lambda)$.

Moreover, the map $\lambda \mapsto T(\lambda)$ is a bijection between Λ and the set of isomorphism classes of indecomposable tilting objects of $\mathcal{C}$.

A standard category is a category satisfying SC1–SC4.

Examples

The following are examples of standard categories:

1. Highest weight categories with finitely many simple objects.
2. Lower finite highest weight categories à la Brundan–Stroppel, e.g. $\mathrm{Rep}(G)$ for a connected reductive group G.
3. The Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$.
4. The category U_q-mod of finite-dimensional type-1 modules for a quantum group U_q associated to a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ and $q \in K$ (+some mild assumptions).
5. $\ldots$

(Generalizing) the AST construction

Inclusions and projections

Let $\mathcal{C}$ be a standard category.

For any $\lambda \in \Lambda$ choose a non-zero morphism

$$c^\lambda \colon \Delta(\lambda) \to \nabla(\lambda)\ .$$

This is unique up to scalars by the Ext-assumption.

Choose an embedding

$$i^\lambda \colon \Delta(\lambda) \hookrightarrow T(\lambda)$$

and a projection

$$\pi^\lambda \colon T(\lambda) \twoheadrightarrow \nabla(\lambda)$$

such that

$$\pi^\lambda \circ i^\lambda = c^\lambda\ .$$

Lifts

Let $T \in \mathcal{C}$ be a tilting object.

Any morphism $f\colon T \to \nabla(\lambda)$ has a lift $\hat{f}\colon T \to T(\lambda)$, i.e.

$$\begin{tikzcd} T \arrow[r, "\hat{f}"] \arrow[dr, "f"'] & T(\lambda) \arrow[d, two heads, "\pi^\lambda"] \\ & \nabla(\lambda) \end{tikzcd}$$

commutes.

Similarly, any morphism $g\colon \Delta(\lambda) \to T$ has a lift $\hat{g}\colon T(\lambda) \to T$, i.e.

$$\begin{tikzcd} T(\lambda) \arrow[r, "\hat{g}"] & T \\ \Delta(\lambda) \arrow[u, hook, "i^\lambda"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

Putting things together

Let $\mathcal{I}_T^\lambda := \{\, 1, \ldots (T : \nabla(\lambda)) \,\}$ and $\mathcal{J}_T^\lambda := \{\, 1, \ldots, (T : \Delta(\lambda)) \,\}$.

Choose a basis $F_T^\lambda = \{f_j \mid j \in \mathcal{J}_T^\lambda\}$ of $\mathrm{Hom}_{\mathcal{C}}(T, \nabla(\lambda))$.

Choose lifts $\hat{F}_T^\lambda := \{\hat{f}_j^\lambda \mid j \in \mathcal{J}_T^\lambda\} \subseteq \mathrm{Hom}_{\mathcal{C}}(T, T(\lambda))$.

Choose a basis $G_T^\lambda = \{g_i^\lambda \mid i \in \mathcal{I}_T^\lambda$ of $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), T)$.

Choose lifts $\hat{G}_T^\lambda := \{\hat{g}_i^\lambda \mid i \in \mathcal{I}_T^\lambda\} \subseteq \mathrm{Hom}_{\mathcal{C}}(T(\lambda), T)$.

Let

$$c_{ij}^\lambda := \hat{g}_i^\lambda \circ \hat{f}_j^\lambda \in \mathrm{End}_{\mathcal{C}}(T)\ .$$

We have

$$\begin{tikzcd}
 & \Delta(\lambda) \arrow[d, hook, "i^\lambda"'] \arrow[dr, "g_i^\lambda"] & \\
T \arrow[r, "\hat{f}_j^\lambda"] \arrow[dr, "f_j^\lambda"'] & T(\lambda) \arrow[r, "\hat{g}_i^\lambda"'] \arrow[d, two heads, "\pi^\lambda"] & T \\
 & \nabla(\lambda) &
\end{tikzcd}$$

Standard bases

Theorem (Bellamy–T.)

$\{c_{ij}^{\lambda} \mid i \in \mathcal{I}_T^{\lambda}, j \in \mathcal{J}_T^{\lambda}\}$ is a standard basis of $E_T := \mathrm{End}_{\mathcal{C}}(T)$.

The key problem is showing that $\{c_{ij}^{\lambda}\}$ is a basis of E_T.

This is where AST use restrictions φ_λ of $\varphi \in E_T$ to weight spaces T_λ of T to obtain a filtration $E_T^{\leq \lambda}$ of E_T, which is then used to prove this.

We replaced:

$$\varphi_\lambda := [\mathrm{Im}\, \varphi \colon L(\lambda)] \in \mathbb{N}$$

and

$$E_T^{\leq \lambda} := \{\varphi \in E_T \mid \varphi_\mu = 0 \text{ unless } \mu \leq \lambda\} .$$

With this replacement, the proof of AST works verbatim (this is not obvious, though).

Duality and cellularity

A problem

An involution on E_T making a standard basis a cellular basis should (philosophically) come from a duality $\mathbb{D}\colon \mathcal{C} \to \mathcal{C}$.

Let $T \in \mathcal{C}$ be a tilting object which is self-dual, i.e. there is

$$\Phi_T\colon \mathbb{D}(T) \xrightarrow{\simeq} T\ .$$

Define a K-algebra anti-morphism $\alpha_T^{-1}\colon E_T \to E_T$ by

$$\alpha_T^{-1}(\varphi) \coloneqq \Phi_T \circ \mathbb{D}(\varphi) \circ \Phi_T^{-1}\ .$$

One computes:

$$\alpha_T^{-2}(\varphi) = a_T \circ \varphi \circ a_T^{-1}\ ,$$

where

$$a_T \coloneqq \Phi_T \circ \mathbb{D}(\Phi^{-1}) \circ \xi_T \in E_T^{\times}\ , \quad \xi\colon \mathrm{id}_{\mathcal{C}} \xrightarrow{\simeq} \mathbb{D}^2\ .$$

Conclusion

1. α^{-1} is an anti-isomorphism.
2. There is no reason why $\alpha^{-2} = \mathrm{id}$!

Standard dualities

Definition

(T, Φ_T) is a fixed point of $\mathbb{D}$ if α_T is an involution, i.e.

$$\Phi_T \circ \mathbb{D}(\Phi^{-1}) \circ \xi_T = \mathrm{id}_T \quad .$$

Definition

$\mathbb{D}$ is a standard duality if it exchanges standard and costandard objects, i.e. $\mathbb{D}(\nabla(\lambda)) \simeq \Delta(\lambda)$, and all indecomposable tilting objects are fixed points of $\mathbb{D}$.

Let $\mathbb{D}$ be a standard duality and $T \in \mathcal{C}$ be a tilting object. Then α_T is an involution on E_T. Choose $\hat{g}_i^\lambda$ as before and define its mirror

$$\hat{f}_i^\lambda := \Phi_{T(\lambda)} \circ \mathbb{D}(\hat{g}_i^\lambda) \circ \Phi_T^{-1} \colon T \to T(\lambda)$$

Let $c_{ij}^\lambda := \hat{g}_i^\lambda \circ \hat{f}_j^\lambda$ for $i, j \in \mathcal{I}_T^\lambda$.

Theorem (Bellamy–T.)

$\{c_{ij}^\lambda \mid i, j \in \mathcal{I}_T^\lambda\}$ is a cellular basis with respect to α_T.

Module dualities

How to check if a duality is a standard duality?

Let A be a K-algebra and suppose $\mathcal{C}$ is a subcategory of A-Mod.

Let τ be an anti-involution on A and consider

$$\mathbb{D} := (-)^{\tau} \circ (-)^{\vee} : A\text{-Mod} \to A\text{-Mod}\ ,$$

where $(-)^{\tau}$ is twist and $(-)^{\vee}$ is a subfunctor of $(-)^{*} = \mathrm{Hom}_K(-, K)$.

Suppose $\mathbb{D}$ restricts to $\mathcal{C}$. We call this a module duality.

Lemma (roughly)

An object T being self-dual under $\mathbb{D}$ is related to the existence of an associative non-degenerate bilinear form on T, and being a fixed point is related to the existence of a symmetric such form.

Idea: To show that a self-dual object T is actually a fixed point, take the symmetrization of the form induced by $\mathbb{D}(T) \simeq T$, and prove it is non-degenerate.

Application: standard and cellular bases on Hecke algebras

Standard bases

Let W be a (finite) complex reflection group and let $\mathcal{H}_{\mathbf{q}}$ be the Hecke algebra (à la Broué, Malle, and Rouquier) of W for an arbitrary parameter $\mathbf{q}$.

Let $H_{\mathbf{c}}$ be the rational Cherednik algebra (à la Etingof and Ginzburg) of W at a "logarithm" $\mathbf{c}$ of $\mathbf{q}$.

There is a category $\mathcal{O}_{\mathbf{c}}$ of $H_{\mathbf{c}}$-modules which is a highest weight category with simple objects indexed by $\mathrm{Irr}(W)$. In particular, $\mathcal{O}_{\mathbf{c}}$ is a standard category.

By Ginzburg, Guay, Opdam, and Rouquier there is a tilting object $T_{\mathbf{c}} \in \mathcal{O}_{\mathbf{c}}$ such that

$$\mathcal{H}_{\mathbf{q}} \simeq \mathrm{End}_{\mathcal{O}_{\mathbf{c}}}(T_{\mathbf{c}}) \ .$$

Theorem (Bellamy–T.)

$\mathcal{H}_{\mathbf{q}}$ has a standard basis coming from the decomposition of $T_{\mathbf{c}}$ into indecomposable tilting objects.

Cellular bases

Suppose that W is a (finite) Coxeter group. Then $\mathcal{O}_{\mathbf{c}}$ is equipped with a module duality $\mathbb{D}$.

Theorem (Bellamy–T.)

$\mathbb{D}$ is a standard duality. In particular, $\mathcal{H}_{\mathbf{q}}$ has a cellular basis.

Problem

Describe the bases and the Specht modules explicitly.

We know that our "co-Specht modules" are isomorphic to the Spect modules of:

Chlouveraki, M., Gordon, I., & Griffeth, S. (2012). Cell modules and canonical basic sets for Hecke algebras from Cherednik algebras.

Furthermore, if W is a Coxeter group and we assume Lusztig's P1–P15, then they are isomorphic to Geck's Specht modules.