THE CENTER OF THE ASYMPTOTIC HECKE CATEGORY AND UNIPOTENT CHARACTER SHEAVES

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OSCAR SYMBOLIC TOOLS

joint work with Liam Rogel

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For me, this started with

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TRUNCATED CONVOLUTION OF CHARACTER SHEAVES

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Abstract

Let G be a reductive, connected algebraic group over an algebraic closure of a finite field. We define a tensor structure on the category of perverse sheaves on G which are direct sums of unipotent character sheaves in a fixed two-sided cell; we show that this is equivalent to the centre with a known monoidal abelian category (a categorification of the J-ring associated to the same two-sided cell).

10. Remarks on the Noncrystallographic Case

10.1 In this subsection we consider a not necessarily crystallographic Coxeter group W' with a fixed two-sided cell $\mathbf{c}'$. The following discussion assumes the truth of Soergel's conjecture for W', recently proved by Elias and Williamson [5]. Let $w \mapsto |w|$ be the length function of W'. For any $w \in W'$ we define $\mathbf{a}(w) \in \mathbf{N}$ as in [19, 13.6]. (The assumption in *loc.cit.* that W' with $w \mapsto |w|$ is bounded in the sense of [19, 13.2] is not necessary for the definition of $\mathbf{a}(w)$; to show that $\mathbf{a}(w)$ is well defined we use instead the inequality $\mathbf{a}(w) \leq |w|$ which is proved by the argument in [19, 15.2], applicable in view of the positivity results of [5].) In the remainder of this section we assume that W' with $w \mapsto |w|$ is bounded; then the properties of $\mathbf{a}(w)$ stated in [19, 14.2] hold by the arguments in [19, §15], using again the positivity results in [5]. Assuming further that W' is either a finite Coxeter group or an affine Weyl group, the ring J and its subring $J^{\mathbf{c}'}$ is defined as in [19, 18.3] in terms of the $\mathbf{a}$-function; both these rings have unit elements.

Pointed out to me by Meinolf Geck in ~2015.

1. Characters of finite reductive groups

Ultimate goal: understand complex here everywhere representations (characters) of finite groups.

Strategy: begin with finite simple groups. These are classified.

Most finite simple groups arise from finite reductive groups.

These are groups like $SL_n(\mathbb{F}_q)$ for powers q of p, which are quasi-simple and yield the finite simple groups $PSL_n(\mathbb{F}_q)$.

The finite reductive groups arise from (infinite) reductive groups G over an algebraic closure $\overline{\mathbb{F}}_p$ as fixed points G^F under a Frobenius map $F: G \to G$, e.g. $G = SL_n(\overline{\mathbb{F}}_p)$ and when F = taking q-th power then $G^F = SL_n(\mathbb{F}_q)$. [We will assume G connected]

This is great! Instead of studying all the G^F separately, we can begin with G and then (try to) transport to G^F. The big G has the advantage of being an algebraic group, so it is a variety and we can use geometric tools!

The big G is also why we may expect some behavior of the G^F independent of q.

even independent of p!

Let W be the Weyl group of G, e.g. $W = S_n$ for $G = SL_n(\overline{\mathbb{F}}_p)$

Deligne–Lusztig theory (1976): construction of virtual characters of G^F via ℓ-adic cohomology of some variety (generalizing Drinfeld's theory for $SL_2(\mathbb{F}_q)$). All irreducible characters appear in them.

Lusztig (1980's): Jordan decomposition of irreducible characters, notions of semisimple and of unipotent characters.

The semisimple characters are "easy", the unipotent ones are hard.

Lusztig (1980s): there is a finite set $\mathcal{U}(W)$ just depending on W which parametrizes the unipotent characters of G^F independently of q!

Moreover, for each $\rho \in \mathcal{U}(W)$ there is a polynomial $\mathrm{Deg}(\rho) \in \mathbb{Q}[\underline{q}]$ (variable) such that when $\underline{q}$ is specialized to q, it gives the degree of the corresponding unipotent representation of G^F.

Observation: W controls the representation theory of the G^F.

Lusztig (1980s): notion of (unipotent) almost characters, which are "explicitly" defined class functions on G^F.

1) Every unipotent character appears in a unipotent almost character
2) The almost characters form an orthonormal basis of the space of class functions on G^F. The base change matrix

$$\left\{ \begin{array}{c} \text{unipotent almost} \\ \text{characters of } G^F \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{unipotent characters} \\ \text{of } G^F \end{array} \right\}$$

is called the Fourier matrix. It is explicitly known. It only depends on W. We denote it by $\mathcal{F}_W$.

Evaluating almost characters (at non-semisimple elements) is hard.

Lusztig's character sheaves (1985): geometric theory (+ categorification) of characters

Fix a prime $\ell \neq p$. The results will not depend on this choice.

Let $D^b_{G,c}(G)$ be the G-equivariant constructible bounded derived category of ℓ-adic sheaves on G.

Character sheaves are certain simple perverse sheaves in $D^b_{G,c}(G)$. There is a notion of unipotent character sheaves.

Let $\mathcal{F}$ be an F-stable character sheaf. Then (by definition) there is an isomorphism $F^*\mathcal{F} \xrightarrow{\phi} \mathcal{F}$. Taking the trace of the induced operator on cohomology (stalks) yields a class function on G^F.

The set of such class functions yields an orthonormal basis for the space of class functions on G^F. They are (in principle) computable!

Lusztig's conjecture (1984):

$$\left\{\begin{matrix}\text{(unipotent) almost} \\ \text{characters}\end{matrix}\right\} \overset{\text{up to scalars}}{=} \left\{\begin{matrix}\text{class functions of } F\text{-stable} \\ \text{(unipotent) character sheaves}\end{matrix}\right\}$$

Shoji (1995): true if G has connected center.

In general the conjecture is still open! Even in the established cases the determination of the scalars is a subtle problem!

Summary:

$$\begin{tikzcd} & & \text{(unipotent) character sheaves} \arrow[d, dashed] \\ \text{(unipotent) irreducible characters of } G^F \arrow[r, leftrightarrow, "F_W"'] & \text{(unipotent) almost characters of } G^F \arrow[r, leftrightarrow, "\text{diagonal matrix (!?)}"'] & \text{characteristic functions} \end{tikzcd}$$

For a detailed introduction see:

1) Carter (2006). A survey on the work of Lusztig
2) Hetz (2023). Characters and character sheaves of finite groups of Lie type. PhD thesis, http://dx.doi.org/10.18419/opus-12932

2. Unipotent character sheaves as a categorical center

$\mathcal{U}_G :=$ subcategory of $D^b_{G,c}(G)$ consisting of direct sums of unipotent character sheaves.

Lusztig (2015): new point of view on $\mathcal{U}_G$ that also categorifies F_W.

$D^b_{B,c}(G/B) :=$ B-equivariant constructible bounded derived category of ℓ-adic sheaves on G/B.

Let $\mathcal{H}_G$ be the geometric Hecke category: subcategory of $D^b_{B,c}(G/B)$ consisting of semisimple perverse sheaves

This is a monoidal category which categorifies the "good old" Iwahori-Hecke algebra $\mathcal{H}_W$:

split Grothendieck group

constant sheaf supported on $\overline{BsB}/B$

$$\mathcal{H}_W \longrightarrow [\mathcal{H}_G]_\oplus \, , \quad b_s \longmapsto [B_s]$$

$\mathbb{Z}[v,v^{-1}]$-algebra isomorphism, $v \mapsto$ shift

Kazhdan-Lusztig's basis

Let c be a two-sided Kazhdan-Lusztig cell of W: recall

$$b_x b_y = \sum_{z\in W} h_{x,y,z} b_z \qquad , \quad h_{x,y,z} \in \mathbb{Z}[v,v^{-1}]$$

Write $z \leq_L y$ if $\exists x \in W$ with $h_{x,y,z} \neq 0$. A left KL cell is an equivalence class of the induced equivalence relation. Analogously, define right KL cells, and then two-sided KL-cells.

two-sided always

Cells help to organize things: to each unipotent character one can associate a unique cell c and one gets a decomposition

$$\mathcal{U}(W) = \bigsqcup_c \mathcal{U}^c(W).$$

This categorifies into a decomposition of $\mathcal{U}_G$ into subcategories $\mathcal{U}_G^c$. Moreover, F_W has block diagonal form with blocks F_W^c indexed by the cells.

Lusztig's asymptotic algebra (J-ring) is the free $\mathbb{Z}$-algebra J with basis $\{j_x \mid x \in W\}$ and multiplication

a-function of W.

$$j_x j_y := \sum_{z\in W} \gamma_{x,y,z^{-1}} j_z \qquad , \quad \gamma_{x,y,z^{-1}} := h_{x,y,z} v^{a(z)}(0) \in \mathbb{Z}$$

Let $J^c := \mathbb{Z}\{j_x \mid x \in c\}$. This is a subalgebra of J and

$$J = \bigoplus_c J^c$$

$\mathcal{H}_G^c :=$ subcategory of $\mathcal{H}_G$ consisting of sheaves supported on c.

Lusztig (1997) constructed a monoidal structure (truncated convolution) on $\mathcal{H}_G^c$ such that

$$J^c \simeq [\mathcal{H}_G^c]_\oplus$$

We call $\mathcal{H}_G^c$ the asymptotic Hecke category.

It follows from Lusztig (1997), Bezrukavnikov-Finkelberg-Ostrik (2009), and Ostrik (2014) that

$\exists\, \Gamma_w^c$ a finite group and Y_w^c a finite Γ_w^c-set, 3-cocycle ω_w^c

$$\mathcal{H}_G^c \simeq \mathrm{Coh}_{\Gamma_w^c}^{\omega_w^c}(Y_w^c \times Y_w^c)$$

Γ_w^c-equivariant coherent sheaves on $Y_w^c \times Y_w^c$ with associator ω_w^c

Hence,

$$\mathcal{Z}(\mathcal{H}_G^c) \simeq \Gamma_w^c\text{-}\mathrm{Vec}_{\Gamma_w^c}^{\omega_w^c}$$

Γ_w^c-equivariant Γ_w^c-graded vector spaces with associator ω_w^c

Here, $\mathcal{Z}$ denotes the categorical (Drinfeld) center: if $\mathcal{C}$ is a monoidal category, then $\mathcal{Z}(\mathcal{C})$ consists of pairs (Z, γ) where $Z \in \mathcal{C}$ and γ is a functorial isomorphism

$$\gamma_X : X \otimes Z \xrightarrow{\simeq} Z \otimes X.$$

has duals

semisimple, rigid, unit simple

in fact modular

The category $\Gamma_w^c\text{-}\mathrm{Vec}_{\Gamma_w^c}^{\omega_w^c}$ is a premodular category: it is fusion, braided (there are functorial isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ $\forall X, Y$), and spherical ("nice" duals).

As such, $\Gamma_w^c\text{-}\mathrm{Vec}_{\Gamma_w^c}^{\omega_w^c}$ has an associated S-matrix

$$S_W^c := \mathrm{Tr}(c_{X,Y} \circ c_{Y,X})$$ for X, Y simple objects

quantum brace

It follows from Lusztig (1984) that

$$S_W^c = F_W^c \quad !$$

The S-matrix is invertible, hence the category is modular (without the "pre")

Is there a geometric interpretation of $\mathcal{Z}(\mathcal{H}_G^c)$?

Lusztig (2015): Yes! There is a monoidal structure on $\mathcal{U}_G^c$ and there is a monoidal equivalence

$$\mathcal{U}_G^c \simeq \mathcal{Z}(\mathcal{H}_G^c) \quad !$$

Conclusion: $\mathcal{U}_G^c$ is a modular category and its S-matrix is equal to the Fourier matrix F_W^c !

3. The non-crystallographic case

Recall:

$$\{\text{finite Coxeter groups}\} = \underset{\underset{\{\text{Weyl groups}\}}{\|}}{\{\text{crystallographic}\}} \,\dot\cup\, \underset{\underset{\text{dihedral groups } I_2(m),\ H_3,\ H_4}{\|}}{\{\text{non-crystallographic}\}}$$

Mysterious: the non-crystallographic groups cannot arise from a reductive group like Weyl groups do. Nonetheless, objects like the Hecke algebra H_W are naturally defined form them as well!

actually equivariant functions on a finite reductive group!

Lusztig (1993): also $U(W)$, $\mathrm{Deg}(\rho)$ can be defined (by ad hoc construction, using H_W)!

Lusztig (1994), Malle (1994), Broué-Malle (1993): ad hoc construction of Fourier matrices F_W!

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TOWARDS SPETSES I

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To Claude Chevalley

Abstract. We present a formalization, using data uniquely defined at the level of the Weyl group, of the construction and combinatorial properties of unipotent character sheaves and unipotent characters for reductive algebraic groups over an algebraic closure of a finite field. This formalization extends to the case where the Weyl group is replaced by a complex reflection group, and in many cases we get families of unipotent characters for a mysterious object, a kind of reductive algebraic group with a nonreal Weyl group, the “spets”.

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The results of Lusztig show that the set of unipotent characters of a “finite reductive group” only depends on the Weyl group of the associated algebraic group together with the action of the Frobenius endomorphism on it. In the course of this classification, Lusztig (see [Lu2]) observed that similar sets can also be attached to those finite real reflection groups which are not Weyl groups (namely, the finite Coxeter groups), just as if there was a “fake algebraic group” whose Weyl group is non-crystallographic. It was at the conference held on the little Greek island named *Spetses*, during the summer of 1993, that we realized that a similar construction might also exist for *non-real* reflection groups.

?

No one knows what a "spets" is.

Note: the geometric Hecke category $\mathcal{H}_G$ is equivalent to the category $\mathcal{H}_W$ of Soergel bimodules (1990, 2007). Like H_W, this category can be naturally defined for any Coxeter group W!

needs Soergel's conjecture
(proven by Elias and Williamson, 2014)

Lusztig (2015): Monoidal structure on $\mathcal{H}_W^c$ (W finite Coxeter group), generalizing the one on $\mathcal{H}_G^c$, categorifying J_W^c.

Conjecture by Lusztig (2015): $\mathcal{U}_W^c := \mathcal{Z}(\mathcal{H}_W^c)$ is a modular category and its S-matrix is equal to F_W^c.

If true, one could view $\mathcal{U}_W^c$ as a category of "unipotent character sheaves on a spets of type W"!

Given the importance of $\mathcal{U}_W^c$ I am so bold to claim:

$\mathcal{U}_W^c$ is the (truncated) spets!

It follows from Elias and Williamson (2021) [relative hard Lefschetz] that $\mathcal{U}_W^c$ is modular.

Theorem (Rogel–Thiel, 2023): Lusztig's conjecture is true for dihedral groups and some (we cannot resolve all) cases of H_3, H_4.

Work in progress of Elias, Rogel, and Tubbenhauer: more cases.

4. Some ideas about the proof

Fabian Mäurer develops software for tensor categories and we found an algorithm for computing the center of a fusion category. (Paper to come)

```
julia

julia> using TensorCategories, Oscar

julia> 𝒞 = I2subcategory(5)
Fusion subcategory of I₂(5)

julia> 𝒵 = Center(𝒞)
Drinfeld center of Fusion subcategory of I₂(5)

julia> simples(𝒵)
4-element Vector{CenterObject}:
 Central object: Bs
 Central object: Bs ⊕ Bsts
 Central object: Bsts
 Central object: Bsts
```

https://fabianmaeurer.github.io/TensorCategories.jl/dev/

https://www.oscar-system.org/

From such experiments we saw what is going on.

1) $\mathcal{H}_W^c$ is multi-fusion in the sense of EGNO: like fusion but the unit object may not be simple

2) $\mathcal{H}_W^c$ has a component subcategory which is fusion

3) Kong and Zheng (2018): If $\mathcal{C}$ is a multifusion category and $\mathcal{C}'$ is a component fusion subcategory, then $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C}')$

4) A component fusion subcategory of $\mathcal{H}_W^c$ is given by the subcategory $\mathcal{H}_W^h$ where h is a diagonal H-cell in c.
(Let c^L be a left cell in c. Then c^L contains a unique Duflo involution by Lusztig (1987). This is contained in $h = c^L \cap c^R$, where c^R is the dual of c^L).

$\boxed{1}$

$s, sts, ststs, \ldots$	$ts, tsts, \ldots$
$st, stst, \ldots$	$t, tst, \ldots$

$\boxed{w_0}$

Situation for dihedral groups

5) $\mathcal{H}_W^h$ is a known fusion category whose center is known as well, e.g. for dihedral groups $\mathcal{H}_W^h \simeq$ even part of the Verlinde category. This is basically in Elias's two-color Soergel calculus (2015).
For H_3, H_4 work of Mackaay, Mazorchuk, Miemietz, Tubbenhauer, Zhang (2023).

6) Carefully piecing everything together (+ work of Lacabanne (2020)).