# INTRODUCTION TO LIE ALGEBRAS AND REPRESENTATION THEORY (FOLLOWING HUMPHREYS)

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***Abstract.*** These are my (personal and incomplete!) notes on the excellent book *Introduction to Lie Algebras and Representation Theory* by Jim Humphreys [1]. Almost everything is basically copied word for word from the book, so ***please do not share this document***.

What is the point of writing these notes then? Well, first of all, they are *my personal* notes and I can write whatever I want. On the more serious side, despite what I just said, I actually tried to spell out most things in my own words, and when I felt I need to give more details (sometimes I really felt it is necessary), I have added more details. My actual motivation is that eventually I want to see what I can shuffle around and add/remove to reflect my personal thoughts on the subject and teach it more effectively in the future.

To be honest, I do not want you to read these notes at all. The book is excellent already, and when you come across something you do not understand, it is much better to struggle with it and try to figure it out by yourself, than coming here and see what I have to say about it (if I have to say anything additionally). Also, these notes are not complete yet, will contain mistakes, and I will not keep it up to date with the current course schedule. It is just *my* notebook for the *future*.

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Recall that $L = \mathfrak{sl}_2$ is the Lie algebra of traceless $(2 \times 2)$-matrices with the commutator as Lie bracket. Its standard basis is

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.1}$$

having commutators

$$[hx] = 2x \ , \quad [hy] = -2y \ , \quad [xy] = h \ . \tag{7.2}$$

**7.1. Weights and maximal vectors.** The matrix $h$ is diagonal, hence this is a semisimple element of $L$, and so by the preservation of the Jordan decomposition it acts diagonally on *any* $L$-module $V$. We thus have a decomposition of $V$ into $h$-eigenspaces

$$V_\lambda = \{v \in V \mid hv = \lambda v\} \tag{7.3}$$

for $\lambda \in F$. Whenever $V_\lambda \neq 0$, we call $\lambda$ a **weight** of $V$ and we call $V_\lambda$ a **weight space**. While $h$ acts by $\lambda$ on $V_\lambda$ by definition, the actions of $x$ resp. $y$ shift weight spaces up and down by 2, resp. More precisely, if $v \in V_\lambda$, then we have

$$h(xv) = [h, x]v + xhv = 2xv + \lambda xv = (\lambda + 2)xv \Longrightarrow xv \in V_{\lambda+2} \ ,$$

and similarly $h(yv) = (\lambda - 2)yv$, so

$$x(V_\lambda) \subseteq V_{\lambda+2} \ , \quad y(V_\lambda) \subseteq V_{\lambda-2} \ . \tag{7.4}$$

From this shifting it follows that $x$ and $y$ act nilpotently on $V$ since this is a finite-dimensional space (this was however already clear from the preservation of the Jordan decomposition since $x$ and $y$ are nilpotent).

Since $V$ is finite-dimensional, there is some $\lambda$ with $V_\lambda \neq 0$ but $V_{\lambda+2} = 0$. We call a corresponding non-zero weight vector $v_0 \in V_\lambda$ a **maximal vector**.

Starting with a maximal vector, we can shift it down repeatedly to lower weight spaces using the action of $y$. More precisely, for $i \geq 0$ let

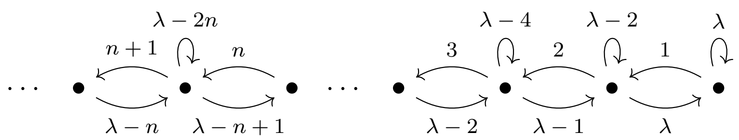
$$v_i := \frac{1}{i!} y^i v_0 \tag{7.5}$$

and set $v_{-1} := 0$. Then you can directly compute that

$$hv_i = (\lambda - 2i)v_i \tag{7.6}$$
$$yv_i = (i+1)v_{i+1} \tag{7.7}$$
$$xv_i = (\lambda - i + 1)v_{i-1} \tag{7.8}$$

From (7.6) it follows that $v_i \in V_{\lambda-2i}$, so we indeed move to lower weight spaces. The $v_i$ all lie in different weight spaces, hence they are *linearly independent*. The formulas above imply that the span of the $v_i$ is stable under $L$, i.e. it is an (non-zero) $L$-submodule of $V$. Moreover, since $V$ is finite-dimensional, there is an index $m$ such that $v_m \neq 0$ but $v_{m+1} = 0$. From (7.7) above it is then clear that $v_{m+j} = 0$ for all $j > 0$. So, the basis forms a *string* $(v_0, \ldots, v_m)$.

**7.2. Classification of irreducible modules.** Now, assume that $V$ is an *irreducible* $L$-module. Since the span of the $v_i$ is a non-zero $L$-submodule of $V$, it must be equal to $V$, hence $(v_0, \ldots, v_m)$ is in fact a *basis* of $V$. Since the $v_i$ all lie in distinct weight spaces, it follows that all weight spaces of $V$ are 1-dimensional. We understand the action of $L$ on $V$ in this basis explicitly due to formulas (7.6)-(7.8), and can depict this as follows (from our book):




(7.9)

The equation (7.8) for the $x$-action reveals a striking fact: for $i = m+1$, the left side is 0, but the right side is $(\lambda - m)v_m$. Since $v_m \neq 0$, we must have

$$\lambda = m = \dim V - 1 \tag{7.10}$$

This has several important implications:

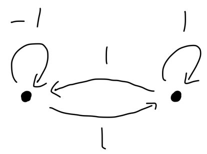
- $\lambda$ is a *natural number*;
- there is a *unique* $\lambda$ with $V_{\lambda+2} = 0$, we call it the **highest weight** of $V$;
- the weights of $V$ are $\lambda, \lambda - 2, \ldots, -\lambda$, hence they are *symmetric* around 0;
- up to scalars there is a *unique* choice of maximal vector;
- up to isomorphism there is a *unique* irreducible $L$-module with highest weight $\lambda$ (for any two modules we go through the construction above and get exactly the same description).

Note that in the last statement above I'm not yet claiming that for *every* natural number $m$ there is an irreducible $L$-module of highest weight $m$. But I'm claiming this now and prove it: take the vector space $V(m)$ with basis

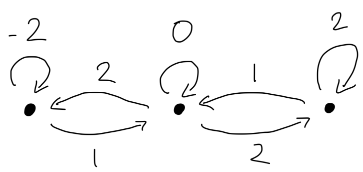
$(v_0, \ldots, v_m)$, set $v_{-1} = 0 = v_{m+1}$ and define an $L$-action on $V(m)$ by the formulas above. This module is irreducible: since the weight spaces of $V(m)$ are by construction just 1-dimensional, any non-zero submodule of $V(m)$ must contain a basis vector $v_i$. And now by the action in the formulas, we can recover every other basis vector $v_j$, i.e. the submodule must be equal to $V(m)$. Hence, $V(m)$ is irreducible. It's clear from the construction that the highest weight of $V(m)$ is equal to $m$. Hence, the irreducible $L$-modules are parametrized by $\mathbb{N}$, and by Weyl's theorem we now have complete knowledge of the category of finite-dimensional $L$-modules. Note that some cases of irreducible modules are familiar:

- $V(0)$ looks like


  So, this is a (the) 1-dimensional module with *trivial* action of $L$.
- $V(1)$ looks like




  This is a 2-dimensional module with basis $(v_0, v_1)$. The $L$-action on these vectors is precisely the matrix multiplication, so $V(1)$ is the *natural representation* of $\mathfrak{sl}_2$.
- $V(2)$ looks like




  This is a 3-dimensional module with basis $(v_0, v_1, v_2)$. When identifying $v_0 = x$, $v_1 = h$, $v_2 = y$, then $V(2) = \mathfrak{sl}_2$ is precisely the *adjoint representation* of $\mathfrak{sl}_2$.

## 8. Root space decomposition

We employ a similar technique as above to study the structure of a general semisimple Lie algebra $L$ via its adjoint representation. Key in the $\mathfrak{sl}_2$-case was the semisimple element $h$, which acts diagonally on any representation. This also exists in general: if $L$ consisted entirely of nilpotent elements, it would be nilpotent by Engel's theorem. This not being the case, we can find $x \in L$ whose semisimple part $x_s$ is non-zero, thus $L$ contains a semisimple element $x_s$. This shows that $L$ possesses non-zero subalgebras consisting entirely of semisimple elements (e.g. the span of $x_s$). Call such a subalgebra **toral**.

### 8.1. Maximal toral subalgebras and roots.

***Lemma 8.1***. *A toral subalgebra is abelian.*

*Proof.* Let $T$ be toral. We have to show that $\operatorname{ad}_T x = 0$ for all $x \in T$. Since $x$ is semisimple, also $\operatorname{ad}_T x$ is semisimple (preservation of the Jordan decomposition), i.e. $\operatorname{ad}_T x$ is diagonalizable. Proving that $\operatorname{ad}_T x = 0$ thus means proving that $\operatorname{ad}_T x$ has no non-zero eigenvalues. So, suppose that it has a non-zero eigenvalue $a$ with eigenvector $y$, i.e. $\operatorname{ad}_T x(y) = ay$. Now, consider $\operatorname{ad}_T y$. This is also diagonalizable since $y$ is semisimple. We can thus write $x = \sum a_\lambda v_\lambda$ as a linear combination of eigenvectors $v_\lambda$ of $\operatorname{ad}_T y$ with eigenvalue $\lambda$. But look what happens when we apply $\operatorname{ad}_T y$. On the one hand we get $\operatorname{ad}_T y(x) = [yx] = -[xy] = -\operatorname{ad}_T(x) = -ay$. This is an eigenvector of $\operatorname{ad}_T y$ of eigenvalue 0. On the other hand, we get $\sum a_\lambda \lambda v_\lambda$. In this sum, only eigenvectors for non-zero eigenvalues are left. Since eigenspaces for distinct eigenvalues are linearly independent, this is impossible—a contradiction. ■

Now, fix a **maximal toral subalgebra** $H$ of $L$ (we will later prove that such subalgebras are unique up to conjugacy in $L$). Since $H$ is abelian, $\operatorname{ad}_L H \subset \mathfrak{gl}(L)$ is a *commuting* family of endomorphisms on $L$. Hence, this family is *simultaneously* diagonalizable, i.e.

$$L = \bigoplus_{\alpha \in H^*} L_\alpha \,, \quad L_\alpha := \{x \in L \mid [hx] = \alpha(h)x \ \forall h \in H\} \,. \tag{8.1}$$

The non-zero $\alpha \in H^*$ with $L_\alpha \neq 0$ are called **roots** and the corresponding spaces $L_\alpha$ are called **root spaces**. The above decomposition of $L$ is called **Cartan decomposition**.

***Example 8.2***. For $L = \mathfrak{sl}_2$ a maximal toral subalgebra $H$ is spanned by $h$. The roots are the linear forms $H \to \mathbb{C}$ mapping $h$ to 2 and $-2$, respectively, and the root spaces are $L_2 = \langle x \rangle$ and $L_{-2} = \langle y \rangle$.

The set $\Phi$ of roots is a discrete finite structure and we will eventually see that it characterizes $L$ completely. This will take some time, so let's first collect some easy facts about root spaces.

***Lemma 8.3***.

*(a)* $[L_\alpha L_\beta] \subset L_{\alpha+\beta}$*;*
*(b)* $L_\alpha$ *for* $\alpha \neq 0$ *consists of ad-nilpotent elements;*
*(c) If* $\alpha + \beta \neq 0$*, then* $L_\alpha \perp L_\beta$ *with respect to the Killing form;*
*(d) The restriction of the Killing form to* $L_0$ *is non-degenerate.*

*Proof.* Let $x \in L_\alpha$, $y \in L_\beta$, and $h \in H$. By the Jacobi identity we have $[h[xy]] + [x[yh]] + [y[hx]] = 0$, i.e. $[h[xy]] = [x[hy]] + [[hx]y] = \beta(h)[xy] + \alpha(h)[xy] =$

$(\alpha+\beta)(h)[xy]$, hence $[xy] \in [L_\alpha L_\beta]$. This proves the first claim. In particular, $[L_\alpha L_\alpha] \subset L_{2\alpha}$, hence $[L_\alpha[L_\alpha L_\alpha]] \subset L_{3\alpha}$ etc. and so, if $\alpha \neq 0$, it is clear that this eventually becomes 0, hence, $L_\alpha$ consists of ad-nilpotent elements. This proves the second claim. For the third, if $\alpha+\beta \neq 0$, there is $h \in H$ such that $(\alpha+\beta)(h) \neq 0$. Take $x \in L_\alpha$ and $y \in L_\beta$. Since the Killing form is associative, we have

$$\alpha(h)\kappa(x,y) = \kappa([hx],y) = -\kappa([xh],y) = -\kappa(x,[hy]) = -\beta(h)\kappa(x,y) \ ,$$

hence $(\alpha+\beta)(h)\kappa(x,y) = 0$. Since $(\alpha+\beta)(h) \neq 0$, this forces $\kappa(x,y) = 0$, proving the third claim. Finally, we know that the Killing form on $L$ is non-degenerate. By the third claim, $L_0$ is orthogonal to any other root space $L_\alpha$. Hence, if $z \in L_0$ is orthogonal to $L_0$, it is orthogonal to all of $L$, hence $z = 0$ since $\kappa$ is non-degenerate. ■

**8.2. Centralizer of $H$.** Notice that

$$L_0 = C_L(H) \tag{8.2}$$

Since $H$ is abelian, we have $H \subseteq C_L(H)$. In fact, we have equality. This is an extremely important result, hence the proof must be a bit more difficult.

***Proposition 8.4.*** *$H = C_L(H)$, i.e. $H$ is* self-centralizing.

*Proof.* Write $C = C_L(H)$. The argumentation goes as follows. We will prove that: (1) $C$ contains the semisimple and nilpotent parts of all its elements; (2) all semisimple elements of $C$ already lie in $H$; (3) $C$ is abelian. Then the assumption $H \neq C$ yields a non-zero nilpotent element $x \in C$. Since $C$ is abelian, the nilpotent operator $\operatorname{ad} x$ commutes with $\operatorname{ad} y$ for $y \in C$, hence also $\operatorname{ad} x \ \operatorname{ad} y$ is nilpotent, hence has trace 0, i.e. $\kappa(x,y) = 0$. This proves that $x$ is in the radical of the restriction of $\kappa$ to $C$. But above we have already proven that this restriction is non-degenerate—a contradiction. So, what remains to be done is proving the three properties of $C$.

(1) Let $x \in C = C_L(H)$. Then $\operatorname{ad} x(H) = 0$. The nilpotent and semisimple parts of $\operatorname{ad} x$ are polynomials in this operator, hence they map $H$ to 0 as well. By the preservation of the Jordan decomposition, it follows that $\operatorname{ad} x_s$ and $\operatorname{ad} x_n$ have the same property, i.e. $x_s, x_n \in C$.

(2) Let $x \in C = C_L(H)$ be semisimple. Since $x$ commutes with $H$ and the sum of commuting semisimple elements is again semisimple, also $H + Fx$ consists of semisimple elements. Since $H$ is maximal, we must have $H = H + Fx$, i.e. $x \in H$ already.

(3) We first show that $C$ is nilpotent. If $x \in C$ is semisimple, then $x \in H$ by (2), so $\operatorname{ad}_C x = 0$, and this is certainly nilpotent. On the other hand, if $x \in C$ is nilpotent, then $\operatorname{ad}_x$ is nilpotent by the preservation of the Jordan decomposition. Now, let $x \in C$ be arbitrary. Then $x = x_s + x_n$ and $x_s, x_n \in C$ by (1). Hence, $\operatorname{ad}_C x$ is the sum of commuting nilpotent operators, hence itself nilpotent. So, every element of $C$ is ad-nilpotent, hence $C$ is nilpotent by Engel's theorem.

Now we prove that $C$ is actually abelian. If not, then $[CC] \neq 0$. Note that $[CC]$ is an ideal, hence a $C$-module. By Engel's theorem, there is $0 \neq x \in [CC]$ which is killed by $C$, which in this case means that $x \in Z(C)$. Hence, $Z(C) \cap [CC] \neq 0$. Take an element $z \neq 0$ from this intersection. We will show that $z$ is not semisimple. Then the nilpotent part $z_n$ of $z$ is non-zero and lies in $C$ by (1). Since $z$ is in the center, it acts trivially on $L$, and since the nilpotent part $\operatorname{ad}_C z_n = (\operatorname{ad}_C z)_n$ is a polynomial in $z$, it acts trivially on $L$ as well, i.e. $z_n \in Z(C)$. So, $\operatorname{ad}_C z_n$ is a nilpotent operator commuting with $\operatorname{ad}_C y$ for $y \in C$, hence $\operatorname{ad}_C z_n \ \operatorname{ad}_C y$ is nilpotent as well, hence has trace 0, i.e. $\kappa(z_n, y) = 0$, so $z \perp C$. This is a contradiction because we have proven above that the restriction of the Killing form to $C$ is non-degenerate.

It remains to show that the element $0 \neq z \in Z(C) \cap [CC]$ is not semisimple. If it would be, it would be contained in $H$ by (2), so $0 \neq z \in H \cap [CC]$. Since $[HC] = 0$ and $\kappa$ is associative, we have $0 = \kappa([HC], C) = \kappa(H, [CC])$, so $\kappa(H, z) = 0$. But this already implies $\kappa(C, z) = 0$: if $x \in C$ is nilpotent, then $\operatorname{ad}_C x$ is a nilpotent operator commuting with $\operatorname{ad}_C z$ (since $z \in H$ by our current assumption and $[HC] = 0$), hence $\operatorname{ad}_C x \ \operatorname{ad}_C z$ is nilpotent, hence has trace 0, i.e. $\kappa(x, z) = 0$. By (1) and (2) this proves that already $\kappa(C, z) = 0$. But this is a contradiction since we have proven above that the restriction of $\kappa$ to $C$ is non-degenerate. ■

An important implication is that the restriction of $\kappa$ to $H$ is *non-degenerate*. This gives us a particular isomorphism between $H^*$ and $H$: to $\alpha \in H^*$ there corresponds an element $t_\alpha \in H$ uniquely characterized by

$$\alpha = \kappa(t_\alpha, -) \ . \tag{8.3}$$

**8.3. Orthogonality properties.** We can now prove further properties about the root space decomposition, generalizing what we we see for $\mathfrak{sl}_2$.

***Proposition 8.5***. *$\Phi$ spans $H^*$. Moreover, for any $\alpha \in \Phi$:*

*(a)* $-\alpha \in \Phi$.
*(b) Let* $x \in L_\alpha$, $y \in L_{-\alpha}$. *Then* $[xy] = \kappa(x, y) t_\alpha$.
*(c)* $[L_\alpha L_{-\alpha}]$ *is one-dimensional with basis* $t_\alpha$.
*(d)* $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.
*(e) Let*

$$h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \ . \tag{8.4}$$

*Then for any non-zero* $x_\alpha \in L_\alpha$ *there is* $y_\alpha \in L_{-\alpha}$ *such that* $x_\alpha$, $y_\alpha$, $h_\alpha = [x_\alpha y_\alpha] = \kappa(x_\alpha, y_\alpha) t_\alpha$ *span a three-dimensional subalgebra of* $L$ *isomorphic to* $\mathfrak{sl}_2$ *via* $x_\alpha \mapsto x$, $y_\alpha \mapsto y$, $h_\alpha \mapsto h$. *We call* $(x_\alpha, h_\alpha, y_\alpha)$ *an* ***$\mathfrak{sl}_2$-triple***.
*(f)* $-h_{-\alpha} = h_\alpha$.

*Proof.* Suppose, $\Phi$ does not span $H^*$. Then by duality, there would be $0 \neq h \in H$ with $\alpha(h) = 0$ for all $\alpha \in \Phi$. But this means, $h$ acts trivially on all root spaces, thus on $L$, i.e. $[hL] = 0$, so $h \in Z(L)$. But $Z(L) = 0$ since $L$ is semisimple—a contradiction.

(a) Suppose, $-\alpha \notin \Phi$. Then $\alpha + \beta \neq 0$ for all $\beta \in \Phi$, hence $L_\alpha \perp L_\beta$ for all $\beta \in \Phi$ by Lemma 8.3. Hence, $L_\alpha \perp L$, contradicting the non-degeneracy of $\kappa$.

(b) Let $h \in H$. Since $\kappa$ is associative, we have $\kappa(h, [xy]) = \kappa([hx], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(\kappa(x, y)t_\alpha, h) = \kappa(h, \kappa(x, y)t_\alpha)$. Hence, $\kappa(h, [xy] - \kappa(x, y)t_\alpha) = 0$, so $H$ is orthogonal to $[xy] - \kappa(x, y)t_\alpha$. Since the restriction of $\kappa$ to $H$ is non-degenerate by above, this forces $[xy] - \kappa(x, y)t_\alpha = 0$, hence the claim.

(c) From (b) it follows that $t_\alpha$ spans $[L_\alpha L_{-\alpha}]$, provided that $[L_\alpha L_{-\alpha}] \neq 0$. To see that this space is indeed non-zero, let $0 \neq x \in L_\alpha$. If $x \perp L_{-\alpha}$, also $x \perp L$, which cannot be true. Hence, there is $0 \neq y \in L_{-\alpha}$ with $\kappa(x, y) \neq 0$. Hence, $[xy] = \kappa(x, y)t_\alpha \neq 0$.

(d) Suppose $\alpha(t_\alpha) = 0$. This means $t_\alpha$ acts by zero on $L_\alpha$, i.e. $[t_\alpha x] = 0$ for all $x \in L_\alpha$. Similarly, as $-\alpha(t_\alpha) = 0$, also $[t_\alpha y] = 0$ for all $y \in L_{-\alpha}$. As in (c) we can find $x, y$ such that $\kappa(x, y) \neq 0$. After rescaling we can assume that $\kappa(x, y) = 1$. Then $[xy] = \kappa(x, y)t_\alpha = t_\alpha$ by (c). By the relations just computed, the subspace $S$ of $L$ spanned by $x, y, h$ is stable under the bracket, hence this is a Lie subalgebra. This algebra is solvable, since the relations show that $[SS] = \langle [xy] \rangle = \langle t_\alpha \rangle$ and $[t_\alpha t_\alpha] = 0$. The derived subalgebra of a solvable algebra is nilpotent by Lie's theorem. Note that the adjoint representation of $L$ is faithful, so $S \simeq \mathrm{ad}_L(S) \subset \mathfrak{gl}(L)$. Since $t_\alpha \in [SS]$ and $[SS]$ is nilpotent, it thus follows that $\mathrm{ad}_L t_\alpha$ is nilpotent. But $\mathrm{ad}_L t_\alpha$ is also semisimple since $t_\alpha \in H$. Hence, $\mathrm{ad}_L t_\alpha = 0$, so $t_\alpha = 0$, a contradiction.

(e) Since $\kappa(x_\alpha, L_{-\alpha}) \neq 0$ and $\kappa(t_\alpha, t_\alpha) \neq 0$, we can find $y_\alpha \in L_{-\alpha}$ such that $\kappa(x_\alpha, y_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$. Then $[x_\alpha y_\alpha] = \kappa(x_\alpha, y_\alpha)t_\alpha = \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha = h_\alpha$. Moreover, $[h_\alpha x_\alpha] = \frac{2}{\kappa(t_\alpha, t_\alpha)}[t_\alpha x_\alpha] = \frac{2}{\alpha(t_\alpha)}\alpha(t_\alpha)x_\alpha = 2x_\alpha$. Similarly, $[h_\alpha y_\alpha] = -2y_\alpha$. So, $x_\alpha$, $y_\alpha$, $h_\alpha$ span a 3-dimensional subalgebra of $L$ with the same multiplication table as $\mathfrak{sl}_2$.

(f) By definition of $t_{-\alpha}$ we have $\kappa(t_{-\alpha}, h) = -\alpha(h)$ for all $h \in H$, hence $\kappa(-t_{-\alpha}, h) = \alpha(h)$, which is the definition of $t_\alpha$. Hence, $-t_{-\alpha} = t_\alpha$, and the relation $-h_{-\alpha} = h_\alpha$ follows. ■

**8.4. Integrality properties.** We can prove further important fact about root spaces using our knowledge of $\mathfrak{sl}_2$-representations. Let $\alpha \in \Phi$ be a root and let $S_\alpha \simeq \mathfrak{sl}_2$ be the subalgebra of an $\mathfrak{sl}_2$-triple for $\alpha$. Clearly, $S_\alpha$ acts on $L$. Recall that $[L_\alpha L_\alpha] \subset L_{2\alpha}$ and $[L_\alpha L_{-\alpha}] \subset H$. Hence, the subspace

$$M = \bigoplus_{c \in F} L_{c\alpha} = H \oplus \bigoplus_{c \in F^*} L_{c\alpha}$$

is an $S_\alpha$-submodule of $L$. We know that $h_\alpha$ acts with weight $2 = \alpha(h_\alpha)$ on $L_\alpha$, so by $2c$ on $L_{c\alpha}$. Hence, the above is already the $\mathfrak{sl}_2$-weight decomposition of $M$ and the weights are $2c$ for $c \in F$ with $L_{c\alpha} \neq 0$. By our knowledge of $\mathfrak{sl}_2$-representations, we know that $\mathfrak{sl}_2$-weights are integral, hence $2c \in \mathbb{Z}$ for all $c$ with $c\alpha \in \Phi$. By Weyl's theorem, $M$ decomposes into a direct sum of irreducible

$\mathfrak{sl}_2$-modules. Irreducible $\mathfrak{sl}_2$-modules having 0 as weight are the $V(m)$ with $m$ even. Any such summand of $M$ must contain a vector of $H$, since this is the 0-weight space of $M$. Now, $U = H + S_\alpha$ is a submodule of $M$. As a vector space we can decompose $H$ into $H = \operatorname{Ker}\alpha \oplus \langle h_\alpha \rangle$, so $U = \operatorname{Ker}\alpha \oplus S_\alpha$ as a vector space. Since $[hx_\alpha]] = \alpha(h)x_\alpha$ and similarly for $y_\alpha$, $S_\alpha$ acts trivially on $\ker\alpha$. Now, $S_\alpha$ is clearly a submodule of $U$, so $S_\alpha$ acts trivially on all other irreducibles in $U$, this means all others are just 1-dimensional and thus contained in $H$. Hence, the only irreducible $\mathfrak{sl}_2$-module in $M$ with even weight spaces is $S_\alpha$, so the only even weights $-2, 0, 2$. It follows that $2\alpha$ cannot be a root since then we would also have weight 4 in $M$. But then $1/2\alpha$ cannot be a root either, so 1 is not a weight of $M$. Hence, there can be no constituents $V(m)$ with $m$ odd in $M$, hence $M = H + S_\alpha$. In particular,

Explain better

$$\dim L_\alpha = 1\ , \tag{8.5}$$

so $S_\alpha$ is uniquely determined as the subalgebra of $L$ generated by $L_\alpha$ and $L_{-\alpha}$. Moreover,

$$\alpha \in \Phi \text{ and } c\alpha \in \Phi \Longrightarrow c = \pm 1\ . \tag{8.6}$$

It remains to study how $S_\alpha$ acts on the other root spaces $L_\beta$ for $\beta \neq \pm\alpha$. Since $[L_\alpha L_\beta] \subset L_{\alpha+\beta}$, it is clear that

$$K = \bigoplus_{i\in\mathbb{Z}} L_{\beta+i\alpha}$$

is an $S_\alpha$-submodule of $L$. The weights are $\beta(h_\alpha) + 2i$, which are obviously all distinct, and these must be integral. In particular,

$$\beta(h_\alpha) \in \mathbb{Z} \tag{8.7}$$

These numbers are called **Cartan integers**. No $\beta + i\alpha$ is equal to 0, so all weight spaces are 1-dimensional by the above. Obviously, not both 0 and 1 can occur as weights of this form. Hence, we can just have a single irreducible $\mathfrak{sl}_2$-module in $K$, i.e. $K$ is irreducible. The highest (resp. lowest) weight must be $\beta(h_\alpha) + 2q$ (resp. $\beta(h_\alpha) - 2r$) if $q$ (resp. $r$) is the largest integer for which $\beta + q\alpha$ (resp. $\beta - r\alpha$) is a root. Moreover, the weights of $K$ form an arithmetic progression with difference 2, so all vectors in the **$\alpha$-string through** $\beta$

$$\beta - r\alpha, \ldots, \beta, \ldots, \beta + q\alpha \tag{8.8}$$

are roots as well, i.e. the string is "unbroken". Notice that the smallest weight is $(\beta - r)(h_\alpha)$ and the largest weight is $(\beta + q)(h_\alpha)$. As the weights in an irreducible $\mathfrak{sl}_2$-module are symmetric, we have $(\beta - r)(h_\alpha) = -(\beta + q)(h_\alpha)$, or

$$\beta(h_\alpha) = \frac{1}{2}(r - q)\alpha(h_\alpha) = r - q\ . \tag{8.9}$$

**8.5. Rationality properties.** Recall that the restriction of the Killing form $\kappa$ to $H$ is non-degenerate. We thus get a particular isomorphism $H^* \to H$ mapping $\beta$ to $t_\beta$, characterized by the property that $\beta = \kappa(t_\beta, -)$. With this isomorphism we can transport $\kappa$ to $H^*$ and get a non-degenerate symmetric bilinear form $(-,-)$ on $H^*$ defined by

$$(\beta,\alpha)=\kappa(t_\beta,t_\alpha)\ . \tag{8.10}$$

Since $\beta=\kappa(t_\beta,-)$, we have $\beta(t_\alpha)=\kappa(t_\beta,t_\alpha)=(\beta,\alpha)$. Recall that $h_\alpha=\frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$, hence we have the expression

$$\mathbb{Z}\ni\beta(h_\alpha)=\frac{2\beta(t_\alpha)}{\kappa(t_\alpha,t_\alpha)}=\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\ . \tag{8.11}$$

for the Cartan integers. We know that $\Phi$ spans $H^*$. Fix a basis $\alpha_1,\dots,\alpha_l$ of $H^*$ consisting of roots. We can then write any other root $\beta\in\Phi$ uniquely as $\sum_{i=1}^l c_i\alpha_i$ with $c_i\in F$. We claim that $c_i\in\mathbb{Q}$. For each $j$ we have

$$(\beta,\alpha_j)=\sum_{i=1}^l c_i(\alpha_i,\alpha_j)\ .$$

Notice that $(\alpha_j,\alpha_j)=\kappa(t_{\alpha_j},t_{\alpha_j})\neq 0$ by what we have proven above. We can thus multiply both sides by $2/(\alpha_j,\alpha_j)$ and get

$$\frac{2(\beta,\alpha_j)}{(\alpha_j,\alpha_j)}=\sum_{i=1}^l\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}c_i\ .$$

This is an *integral* (in particular *rational*) linear system in $l$ equations with $l$ unknowns $c_i$. Since $(\alpha_1,\dots,\alpha_l)$ is a basis and the form is non-degenerate, the Gram matrix $((\alpha_i,\alpha_j))_{ij}$ is non-degenerate, and so the same is true for the coefficient matrix of this system. Hence, the system has a unique solution over $\mathbb{Q}$, so $c_i\in\mathbb{Q}$ for all $i$ as claimed.

Let $E_\mathbb{Q}$ be the $\mathbb{Q}$-subspace of $H^*$ spanned by all the roots. We have just proven that

$$\dim_\mathbb{Q}E_\mathbb{Q}=l=\dim_F H^*\ . \tag{8.12}$$

We claim that we can also restrict $(-,-)$ to $E_\mathbb{Q}$. For $\lambda,\mu\in H^*$ we have

$$(\lambda,\mu)=\kappa(t_\lambda,t_\mu)=\mathrm{Tr}(\mathrm{ad}\,t_\lambda\ \mathrm{ad}\,t_\mu)=\sum_{\alpha\in\Phi}\alpha(t_\lambda)\alpha(t_\mu)=\sum_{\alpha\in\Phi}(\alpha,\lambda)(\alpha,\mu)\ .$$

In particular, we have

$$(\beta,\beta)=\sum_{\alpha\in\Phi}(\alpha,\beta)^2\ .$$

Dividing by $(\beta,\beta)^2\neq 0$, we get

$$\frac{1}{(\beta,\beta)}=\sum_{\alpha\in\Phi}\frac{(\alpha,\beta)^2}{(\beta,\beta)^2}=\sum_{\alpha\in\Phi}\left(\frac{(\alpha,\beta)}{(\beta,\beta)}\right)^2\in\mathbb{Q}$$

by above. Hence, $(\beta,\beta)\in\mathbb{Q}$. Then also $(\beta,\alpha)=\beta(h_\alpha)(\alpha,\alpha)\in\mathbb{Q}$. Hence, we can restrict $(-,-)$ to a rational form on $E_\mathbb{Q}$. Even more, this is in fact positive definite by the equation above (sum of squares), hence it is an *inner product* on $E_\mathbb{Q}$.

**8.6. Example: Root system for $\mathfrak{sl}_n$.** Let's first look at $\mathfrak{sl}_2$ with standard basis $x, h, y$ as usual. The Gram matrix of the Killing form in this basis is

$$\kappa = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix} . \tag{8.13}$$

We have $H = \langle h \rangle$, so a 1-dimensional space. We have two roots $\pm\alpha$, where $\alpha \colon H \to \mathbb{C}$ is the map $h \mapsto 2$ (recall that $h$ acts by 2 on $x$). Recall that $t \coloneqq t_\alpha \in H$ is the element satisfying $\alpha = \kappa(t, -)$. It is enough to check this on $h$, so $2 = \alpha(h) = \kappa(t, h)$. Hence,

$$t = \frac{1}{4}h . \tag{8.14}$$

For the inner product $(-,-)$ on $H^*$ we get

$$(\alpha, \alpha) = \kappa(t,t) = \kappa(\frac{1}{4}h, \frac{1}{4}h) = \frac{8}{16} = \frac{1}{2} . \tag{8.15}$$

Hence, our root vector $\alpha$ has length $1/\sqrt{2}$.

Now, we're doing $\mathfrak{sl}_n$ in general. The standard basis of $\mathfrak{sl}_n$ are the elementary matrices $e_{ij}$ for $1 \leq i \neq j \leq n$ together with the diagonal matrices $h_i \coloneqq e_{ii} - e_{i+1,i+1}$ for $1 \leq i < n$. There are $\sum_{i=1}^{n-1} i = \frac{1}{2}(n-1)n$ vectors $e_{ij}$ with $i < j$, so $(n-1)n$ vectors $e_{ij}$ with $i \neq j$; together with $n-1$ vectors $h_i$ this makes a total of $n(n-1) + (n-1) = n^2 - 1$ vectors. The $h_i$ span a maximal toral subalgebra $H \subset \mathfrak{sl}_n$. Recall the commutator relation

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} . \tag{8.16}$$

Define $\epsilon_i \colon H \to \mathbb{C}$ by

$$\epsilon_i(h_k) \coloneqq \delta_{ik} - \delta_{i,k+1} \tag{8.17}$$

and

$$\alpha_{ij} \coloneqq \epsilon_i - \epsilon_j \tag{8.18}$$

Clearly, $\alpha_{ji} = -\alpha_{ij}$. Setting

$$\alpha_i \coloneqq \epsilon_i - \epsilon_{i+1} \tag{8.19}$$

we have for $i < j$ the expression

$$\alpha_{ij} = \alpha_i + \ldots + \alpha_{j-1} .$$

We have

$$\begin{aligned} [h_k, e_{ij}] &= [e_{kk} - e_{k+1,k+1}, e_{ij}] = [e_{kk}, e_{ij}] - [e_{k+1,k+1}, e_{ij}] \\ &= \delta_{ki} e_{kj} - \delta_{jk} e_{ik} - \delta_{k+1,i} e_{k+1,j} + \delta_{j,k+1} e_{i,k+1} \\ &= (\delta_{ik} - \delta_{i,k+1} - \delta_{jk} + \delta_{j,k+1}) e_{ij} \\ &= \alpha_{ij}(h_k) e_{ij} , \end{aligned}$$

hence $H$ acts via $\alpha_{ij}$ on $e_{ij}$. It follows that the $\alpha_{ij}$ are roots with root space spanned by $e_{ij}$. These must be all roots already by dimension reasons. Hence,

$$\Phi = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\} . \tag{8.20}$$

For $i < j$ we define

$$x_{ij} \coloneqq e_{ij} , \quad y_{ij} \coloneqq e_{ji} , \quad h_{ij} \coloneqq h_i + \ldots + h_{j-1} . \tag{8.21}$$

We have

$$[x_{ij}, y_{ij}] = [e_{ij}, e_{ji}] = \delta_{jj} e_{ii} - \delta_{ii} e_{jj} = e_{ii} - e_{jj} = h_i + \ldots + h_{j-1} = h_{ij} \tag{8.22}$$

and

$$\begin{aligned} [h_{ij}, x_{ij}] &= \sum_{k=i}^{j-1} [h_k, e_{ij}] = \sum_{k=i}^{j-1} \alpha_{ij}(h_k) e_{ij} = \sum_{k=i}^{j-1} (\epsilon_i - \epsilon_)(h_k) e_{ij} \\ &= \sum_{k=i}^{j-1} (\delta_{ik} - \delta_{i,k+1} - \delta_{jk} + \delta_{j,k+1}) e_{ij} = (\delta_{ii} - \delta_{ij} - \delta_{ji} + \delta_{jj}) e_{ij} \\ &= 2e_{ij} = 2x_{ij} . \end{aligned}$$

Similarly,

$$[h_{ij}, y_{ij}] = -2y_{ij} . \tag{8.23}$$

Hence, $(x_{ij}, h_{ij}, y_{ij})$ is an $\mathfrak{sl}_2$-triple. We want to compute the inner product on $H^*$, so we need to compute the Killing form. This is a bit annoying. It's not too difficult to show that $\kappa$ is actually related to the usual matrix trace by $\kappa = 2n\mathrm{Tr}$. This makes the computation simpler but I'll do it here directly once and just calculate the hell out of it. By our general results, the vectors $t_{ij} \coloneqq t_{\alpha_{ij}}$ with the property $\kappa(t_{ij}, -) = \alpha_{ij}$ satisfy

$$[x_{ij} y_{ij}] = \kappa(x_{ij}, y_{ij}) t_{ij} , \tag{8.24}$$

so we want to compute

$$\kappa(x_{ij}, y_{ij}) = \kappa(e_{ij}, e_{ji}) = \mathrm{Tr}(\mathrm{ad}\, e_{ij} \ \ \mathrm{ad}\, e_{ji}) . \tag{8.25}$$

We need to express the operator $\mathrm{ad}\, e_{ij} \ \ \mathrm{ad}\, e_{ji}$ in our standard basis of $\mathfrak{sl}_n$. For basis elements $e_{kl}$ with $k \neq l$ we have

$$\begin{aligned} (\mathrm{ad}\, e_{ij}\, \mathrm{ad}\, e_{ji})(e_{kl}) &= [e_{ij}, [e_{ji}, e_{kl}]] = \delta_{ik} e_{il} - \delta_{ik}\delta_{li} e_{jj} - \delta_{lj}\delta_{jk} e_{ii} + \delta_{lj} e_{kj} \\ &= (\delta_{ik} + \delta_{lj}) e_{kl} + \text{ basis elements other than } e_{kl} . \end{aligned}$$

Going through all $k \neq l$, the diagonal coefficients sum up to $2(n-1)$. For basis elements $h_k$ we get

$$\begin{aligned} (\mathrm{ad}\, e_{ij}\, \mathrm{ad}\, e_{ji})(h_k) &= [e_{ij}, [e_{ji}, h_k]] = [e_{ij}, -[h_k, e_{ji}]] = [e_{ij}, -\alpha_{ji}(h_k) e_{ji}] \\ &= -\alpha_{ji}(h_k)[e_{ij}, e_{ji}] = \alpha_{ij}(h_k)[e_{ij}, e_{ji}] = \alpha_{ij}(h_k)(\delta_{jj} e_{ii} - \delta_{ii} e_{jj}) \\ &= \alpha_{ij}(h_k)(h_{ij}) = (\epsilon_i - \epsilon_j)(h_k)(h_i + \ldots h_{j-1}) \\ &= (\delta_{ik} - \delta_{i,k+1} - \delta_{j,k} + \delta_{j,k+1})(h_i + \ldots + h_{j-1}) \\ &= (\delta_{ik} + \delta_{j,k+1})(h_i + \ldots + h_{j-1}) = 2h_k + \text{ other basis vectors} , \end{aligned}$$

the equation before the last being due to $i \leq k \leq j-1$. So we get an additional 2 for the trace, in total

$$\kappa(x_{ij}, y_{ij}) = 2(n-2) + 2 = 2n\ . \tag{8.26}$$

Now,

$$h_{ij} = [x_{ij}, y_{ij}] = \kappa(x_{ij}, y_{ij}) t_{ij} = 2n t_{ij}\ ,$$

so

$$t_{ij} = \frac{1}{2n} h_{ij}\ . \tag{8.27}$$

Hence, for the scalar product on $H^*$ we get

$$(\alpha_{ij}, \alpha_{kl}) = \kappa(t_{ij}, t_{kl}) = \frac{1}{4n^2}\kappa(h_{ij}, h_{kl}) \tag{8.28}$$

Computing this is even more annoying. First, from our generalities we know that

$$h_{ij} = \frac{2}{\kappa(t_{ij}, t_{ij})} t_{ij}\ .$$

Since $h_{ij} = 2n t_{ij}$, we conclude that $2n = 2/\kappa(t_{ij}, t_{ij})$, so

$$(\alpha_{ij}, \alpha_{ij}) = \frac{1}{n}\ , \tag{8.29}$$

hence

$$\|\alpha_{ij}\| = \frac{1}{\sqrt{n}}\ . \tag{8.30}$$

In particular, all roots have the *same* length. What about the other scalar products (I want to know the angle between root vectors)? Since the Killing form is bilinear and symmetric, it's enough to look at $(\alpha_i, \alpha_j) = \frac{1}{4n^2}\kappa(h_i, h_j)$ for $i < j$. This means we have to compute the trace of the operator $\operatorname{ad} h_i\ \operatorname{ad} h_j$. This acts trivially on $H$, so we only need to look at basis elements $e_{kl}$. We can assume $k < l$; the case $l > k$ gives in the trace another equal contribution. So,

$$\begin{aligned}
\operatorname{ad} h_i \operatorname{ad} h_j(e_{kl}) &= \alpha_{kl}(h_i)\alpha_{kl}(h_j) e_{kl} = \left((\epsilon_k - \epsilon_l)(h_i) \cdot (\epsilon_k - \epsilon_l)(h_j)\right) e_{kl} \\
&= (\delta_{ki} - \delta_{k,i+1} - \delta_{li} + \delta_{l,i+1})(\delta_{kj} - \delta_{k,j+1} - \delta_{lj} + \delta_{l,j+1}) \\
&= \delta_{ki}\delta_{kj} - \delta_{ki}\delta_{k,j+1} - \delta_{ki}\delta_{lj} + \delta_{ki}\delta_{l,j+1} \\
&\quad - \delta_{k,i+1}\delta_{kj} + \delta_{k,i+1}\delta_{k,j+1} + \delta_{k,i+1}\delta_{lj} - \delta_{k,i+1}\delta_{l,j+1} \\
&\quad - \delta_{li}\delta_{kj} + \delta_{li}\delta_{k,j+1} + \delta_{li}\delta_{lj} - \delta_{li}\delta_{l,j+1} \\
&\quad + \delta_{l,i+1}\delta_{kj} - \delta_{l,i+1}\delta_{k,j+1} - \delta_{l,i+1}\delta_{lj} + \delta_{l,i+1}\delta_{l,j+1}
\end{aligned}$$

Now, a few terms disappear immediately because of $i < j$ and $k < l$:

$$= -\delta_{ki}\delta_{lj} + \delta_{ki}\delta_{l,j+1} - \delta_{k,i+1}\delta_{kj} + \delta_{k,i+1}\delta_{lj} - \delta_{k,i+1}\delta_{l,j+1} - \delta_{l,i+1}\delta_{lj}\ .$$

Let's first look at the special case $j = i+1$. We then get

$$= -\delta_{ki}\delta_{l,i+1} + \delta_{ki}\delta_{l,i+2} - \delta_{k,i+1} - \delta_{k,i+1}\delta_{l,i+2} - \delta_{l,i+1}\ .$$

For the trace of $\operatorname{ad} h_i \operatorname{ad} h_j$ we sum up over all $k < l$ (and then double for $k > l$). Summing the above over all $k < l$ gives $-1 + 1 - (n - (i+2) + 1) - 1 - i = -n$. Hence,

$$\kappa(h_i, h_{i+1}) = -2n \ , \tag{8.31}$$

so

$$(\alpha_i, \alpha_{i+1}) = \frac{1}{4n^2}\kappa(h_i, h_{i+1}) = \frac{-2n}{4n^2} = -\frac{1}{2n} \ . \tag{8.32}$$

Hence, for the angle $\theta$ between $\alpha_i$ and $\alpha_{i+1}$ we get

$$\cos\theta = \frac{(\alpha_i, \alpha_{i+1})}{\|\alpha_i\| \, \|\alpha_{i+1}\|} = -\frac{1}{2n} n = -\frac{1}{2} \Longrightarrow \theta = \frac{2}{3}\pi = 120° \ . \tag{8.33}$$

Whenever $j \neq i + 1$ you can see from the above equations that

$$\kappa(h_i, h_j) = 0 \ , \tag{8.34}$$

i.e. $\alpha_i$ and $\alpha_j$ are orthogonal.

Conclude that $\mathfrak{sl}_n$ is semisimple

So, for $\mathfrak{sl}_3$ we get the following picture that you should remember well:

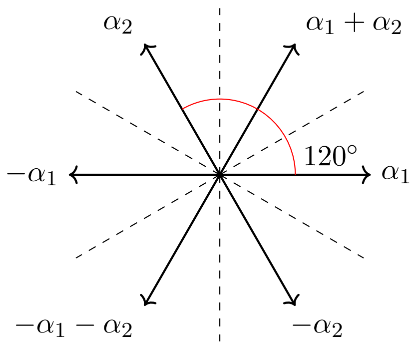



***Figure 8.1.*** Roots for $\mathfrak{sl}_3$.

Root systems for other types

# CHAPTER III

# ROOT SYSTEMS

Let's wrap up what we just did. In a semisimple Lie algebra $L$ we picked a maximal toral subalgebra $H$. The weights for the action of $H$ on $L$ gave us a finite subset $\Phi \subset H^*$ (called roots). They span a real vector space $E$ in $H^*$ of dimension equal to the complex dimension of $H^*$. The restriction of the Killing form $\kappa$ to $H$ is non-degenerate, thus induces a non-degenerate form on $H^*$. The form on the roots is actually $\mathbb{Q}$-valued and it restricts to an inner product on $E$. We thus have produced from $L$ a finite *vector configuration* in a euclidean space $E$. Think of this as a *combinatorial footprint* of $L$. We have already established the following properties about $\Phi$:

(R1) $\Phi$ is finite, spans $E$ and $0 \notin \Phi$,
(R2) If $\alpha \in \Phi$, the only multiples of $\alpha \in \Phi$ are $\pm\alpha$,
(R3) If $\alpha, \beta \in \Phi$, then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \beta(h_\alpha)\alpha \in \Phi$,
(R4) If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta(h_\alpha) \in \mathbb{Z}$ (Cartan integers).

Property (R3) says in fact that $\Phi$ is *symmetric under reflections* in the vectors of $\Phi$. We will now make precise what this means. It will be useful to study abstractly a vector configuration $\Phi$ in a euclidean space satisfying the above properties.

## 9. Axiomatics

**9.1. Reflections in a euclidean space.** Throughout, let $E$ be a euclidean vector space, i.e. a finite-dimensional vector space over $\mathbb{R}$ equipped with an inner product $(-,-)$. The key example is $H$ with the inner product induced by the Killing form. A **reflection** in $E$ is a linear automorphism $\sigma$ on $E$ fixing pointwise a hyperplane $P$ and sending any vector orthogonal to that hyperplane to its negative. Note that $E = P \oplus P^\perp$ and that $P^\perp$ is 1-dimensional. It is easy to write down an explicit formula for $\sigma$: pick a non-zero vector $\alpha \in P^\perp$; then

$$\sigma = \sigma_\alpha := \beta \mapsto \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \, . \tag{9.1}$$

This is because $\sigma_\alpha(\beta) = \beta$ for $\beta \in \alpha^\perp = P$ and $\sigma_\alpha(\alpha) = -\alpha$, and this is exactly what $\sigma$ is doing. From this formula you can see that a reflection is an *orthogonal transformation*. Note that any $\alpha \in E$ defines a reflection $\sigma_\alpha$ by the above formula fixing the hyperplane $P_\alpha := \alpha^\perp$, and that $\sigma_{c\alpha} = \sigma_\alpha$ for any non-zero scalar $c$. Also note that a reflection is an *involution*, i.e. of order 2. To simplify notations in the following, we set

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \, . \tag{9.2}$$

These numbers are called **Cartan integers**.

**9.2. Root systems.** A set $\Phi$ in a euclidean space $E$ satisfying (R1) to (R4) is called a **root system**. The dimension of $E$ is called the **rank** of the root system. From the formula for reflections we see that the third property on $\Phi$ actually means that the reflection $\sigma_\alpha$ preserves $\Phi$, i.e. $\Phi$ is "symmetric" under reflections in its vectors. Let $W$ be the subgroup of $\mathrm{GL}(E)$ generated by the $\sigma_\alpha$ for $\alpha \in \Phi$. This "symmetry group" of $\Phi$ is called the **Weyl group** of $\Phi$. By what we just said, any $\sigma_\alpha$ permutes the roots, so defines a unique element in the symmetric group $S_\Phi$. Since $\Phi$ spans $E$, this permutation uniquely characterizes $\sigma_\alpha$, so we have an injective group morphism $W \to S_\Phi$ and we can thus identify $W$ with a subgroup of $S_\Phi$. Since $\Phi$ is finite, the group $S_\Phi$ is finite, hence $W$ is *finite*. The Weyl group describes "reflection symmetries" of $\Phi$. When we say "symmetry" we actually mean *automorphism* of a root system. But we have not yet defined what this actually is. Here's the definition given in [1].

***Definition 9.1.*** An **isomorphism** of root systems $(E, \Phi)$ and $(E', \Phi')$ is a vector space isomorphism (not necessarily an isometry) $f\colon E \to E'$ with $f(\Phi) = \Phi'$ and preserving the Cartan integers, i.e. $\langle f(\alpha), f(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$.

In particular, we now have the notion of an **automorphism** of a root system $(E, \Phi)$. We denote the group of automorphisms by $\mathrm{Aut}(E, \Phi)$. With the same argumentation as above for the Weyl group we have an injective map $\mathrm{Aut}(E, \Phi) \to S_\Phi$ into the symmetric group on $\Phi$, hence $\mathrm{Aut}(E, \Phi)$ is a *finite* group. Clearly, any orthogonal map (isometry) preserving the roots also preserves the Cartan integers. In particular, $W$ is a subgroup of $\mathrm{Aut}(E, \Phi)$.

***Remark 9.2.*** I feel Definition 9.1 needs some clarification and motivation. This is not done so well in [1] and I hope the following helps. It makes sense to first look at an isomorphism of Lie algebras and see what it does with the root systems. The answer is a straightforward "basically nothing" but I'll do the computation anyways to explain a subtle point.

So, let $\phi\colon L \to L'$ be an isomorphism of semisimple Lie algebras. The image $H' := \phi(H)$ of our maximal toral subalgebra $H \subset L$ is a maximal toral subalgebra of $L'$ and we get a corresponding root system $\Phi'$ in $E' = \mathbb{R}\Phi' \subset (H')^*$. Note that if $\alpha\colon H \to \mathbb{C}$ is a linear form on $H$, then $\alpha \circ \phi^{-1}\colon H' \to \mathbb{C}$ is a linear form on $H'$. This yields an isomorphism $f\colon H^* \to (H')^*$. What happens to the roots and to the root spaces? Let $h' = \phi(h) \in H'$ and $x' = \phi(x) \in L$. Then

$$[h'x'] = [\phi(h)\phi(x)] = \phi([hx]) \, .$$

So, if $\alpha \in \Phi$ and $x \in L_\alpha$, then

$$[h'x'] = \phi(\alpha(h)x) = \alpha(h)\phi(x) = \alpha \circ \phi^{-1}(h')x' = f(\alpha)(h')x' \ ,$$

i.e., $x' \in L'_{f(\alpha)}$. This shows that $f(L_\alpha) = L'_{f(\alpha)}$ and that $f$ induces a bijection

$$f \colon \Phi \xrightarrow{1:1} \Phi' \ . \tag{9.3}$$

Consequently, $f \colon H^* \to (H')^*$ restricts to an isomorphism $\mathbb{R}\Phi = E \to E' = \mathbb{R}\Phi'$. What happens to the inner product? Let $x' = \phi(x), y' = \phi(y) \in L'$. Then

$$\operatorname{ad}_{L'} x'(y') = [x'y'] = [\phi(x)\phi(y)] = \varphi([xy]) = \phi(\operatorname{ad}_L x(y)) \ ,$$

so the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\operatorname{ad}_L} & \mathfrak{gl}(L) \\ \phi\downarrow & & \downarrow A\mapsto\phi\circ A\circ\phi^{-1}V \\ L' & \xrightarrow[\operatorname{ad}_{L'}]{} & \mathfrak{gl}(L') \end{array}$$

Hence, for the Killing form $\kappa'$ on $L'$ we get

$$\kappa'(x', y') = \operatorname{Tr}(\operatorname{ad}_{L'} x' \operatorname{ad}_{L'} y') = \operatorname{Tr}(\operatorname{ad}_L x \operatorname{ad}_L y) = \kappa(x, y) \ .$$

Recall the isomorphism $H^* \to H$ induced by $\kappa$ mapping $\alpha$ to $t_\alpha$ characterized by the property that $\alpha = \kappa(t_\alpha, -)$. Similarly, we have an isomorphism $(H')^* \to H'$ induced by $\kappa'$. Now,

$$\begin{aligned} \kappa'(t_{f(\alpha)}, -) &= f(\alpha) = \alpha \circ \phi = \kappa(t_\alpha, \phi^{-1}(-)) = \kappa(\phi^{-1}(f(t_\alpha)), \phi^{-1}(-)) \\ &= \kappa'(\phi(t_\alpha), -) \ , \end{aligned}$$

hence

$$t_{f(\alpha)} = \phi(t_\alpha) \ ,$$

so the diagram

$$\begin{array}{ccc} H^* & \xrightarrow{\alpha\mapsto t_\alpha} & H \\ f\downarrow & & \downarrow\phi \\ (H')^* & \xrightarrow[\alpha'\mapsto t_{\alpha'}]{} & H' \end{array} \tag{9.4}$$

commutes. Hence, for the scalar product $(-,-)'$ on $H'$ induced by $\kappa'$ we have

$$(\alpha', \beta')' = \kappa'(t_{\alpha'}, t_{\beta'}) = \kappa'(\phi(t_\alpha), \phi(t_\beta)) = \kappa(t_\alpha, t_\beta) = (\alpha, \beta) \ .$$

Hence, the isomorphism $f \colon E \to E'$ is an *orthogonal* transformation.

To sum up, an isomorphism $\phi \colon L \to L'$ of Lie algebras induces an orthogonal transformation $f \colon E \to E'$ between the spaces where the root systems $\Phi$ and $\Phi'$ live in such that $f(\Phi) = \Phi'$. So, this would be a natural definition of an *isomorphism* of root systems, right? Notice that Definition 9.1 is more general: orthogonality was relaxed to preservation of Cartan integers only. I want to argue why this is necessary. Suppose $L$ is a semisimple Lie algebra such that $E$, and thus $H$, is 1-dimensional. Then there's only one root and its negative, so $L$ is 3-dimensional and must be isomorphic to $\mathfrak{sl}_2$. Hence, up to isomorphism there's only *one* such Lie algebra. But if we include orthogonality in the notion of isomorphism of root systems there would be *infinitely* many non-isomorphic one-dimensional root systems: any scaling would yield another root system! So, we would get way too many root systems, only *one* actually comes from a Lie algebra! To connect *abstract* root systems back to the *concrete* Lie theory context, we need to relax the orthogonality condition—what was done in Definition 9.1. But why is it natural to assume that the Cartan integers should be preserved? We will see later that the Cartan integers actually pin down the isomorphism class of a Lie algebra, they are *structure constants*. So, with the notion of isomorphisms in Definition 9.1 we will get a bijection between isomorphism classes of semisimple Lie algebras and isomorphism classes of root systems (we still need to prove this of course).

What is a bit surprising (to me) is that in fact *any* vector space isomorphism preserving the roots already *automatically* preserves the Cartan integers, so we could have excluded this from the definition! This is the essence of the two lemmas in [1, §9.1, §9.2] to which I am coming now.

***Lemma 9.3***. *Let $(E,\Phi)$ be a root system and let $\sigma \in \mathrm{GL}(E)$ be a reflection with $\sigma(\Phi) = \Phi$ and mapping some $\alpha \in \Phi$ to its negative. Then already $\sigma = \sigma_\alpha$.*

*Proof.* Let $\tau = \sigma\sigma_\alpha$. We need to show that $\tau = \mathrm{id}$. We clearly have $\tau(\alpha) = \alpha$. Moreover, $\tau$ also acts as the identity on $E/\langle\alpha\rangle$. Here's why.[1] Let $P$ be the hyperplane fixed by $\sigma$ and $P_\alpha$ the one fixed by $\sigma_\alpha$. Since $\sigma(\alpha) = -\alpha$, we have $\alpha \notin P$. We thus have two decompositions $E = \mathbb{R}\alpha \oplus P_\alpha = \mathbb{R}\alpha \oplus P$. Take a basis $\alpha, \alpha_1, \ldots, \alpha_{l-1}$ of $E$. We can then write $\alpha_i = \beta_i + b_i\alpha = \gamma_i + a_i\alpha$ with $\beta_i \in P_\alpha$, $\gamma_i \in P$, and $a_i, b_i \in \mathbb{R}$. We then get

$$\begin{aligned}\tau(\alpha_i) &= \sigma\sigma_\alpha(\alpha_i) = \sigma\sigma_\alpha(\beta_i + b_i\alpha) = \sigma(\beta_i - b_i\alpha) = \sigma(\gamma_i + a_i\alpha - 2b_i\alpha) \\ &= \gamma_i - a_i\alpha + 2b_i\alpha = \alpha_i + 2(b_i - a_i)\alpha \ .\end{aligned}$$

Hence, $\tau(\alpha_i) \equiv \alpha_i \bmod \mathbb{R}\alpha$. This also holds for $\alpha$, so $\tau \equiv \mathrm{id} \bmod \mathbb{R}\alpha$. Hence, $\tau$ acts trivially on $\mathbb{R}\alpha$ and on $E/\mathbb{R}\alpha$. So, all eigenvalues of $\tau$ are 1, and so the minimal polynomial of $\tau$ divides $(T-1)^l$. Since $\Phi$ is finite and $\tau(\Phi) = \Phi$, not all vectors $\beta, \tau(\beta), \tau^2(\beta), \ldots$ can be distinct. Hence, there is $k$ such that $\tau^k(\beta) = \beta$ for all $\beta \in \Phi$. Because $\Phi$ spans $E$, this forces $\tau^k = \mathrm{id}$, so the minimal polynomial of $\tau$ divides $T^k - 1$. Combined, the minimal polynomial of $\tau$ divides $\gcd((T-1)^l, T^k - 1) = T - 1$, i.e. $\tau = \mathrm{id}$. ■

[1] It took me a few hours to deduce this triviality. Ugh. Sometimes, knowing that something is trivial is just not enough...

***Lemma 9.4.*** *Let $(E, \Phi)$, $(E', \Phi')$ be root systems and let $f\colon E \to E'$ be a vector space isomorphism with $f(\Phi) = \Phi'$. Then $f$ is already an isomorphism $(E, \Phi) \to (E, \Phi')$ of root systems, i.e. it preserves the Cartan integers. Moreover:*

*(a)* $\sigma_{f(\alpha)} = f \circ \sigma_\alpha \circ f^{-1}$ *for all* $\alpha \in \Phi$.

*(b)* $f$ *preserves* angles *between any two roots and* ratios *of lengths between any two non-orthogonal roots.*

*Proof.* The key is property (a). Assume we can show this holds. By definition of the root reflections we have the relations

$$f(\sigma_\alpha(\beta)) = f(\beta - \langle \alpha, \beta \rangle \alpha) = f(\beta) - \langle \alpha, \beta \rangle f(\alpha) \tag{9.5}$$

and

$$\sigma_{f(\alpha)}(f(\beta)) = f(\beta) - \langle f(\alpha), f(\beta) \rangle f(\alpha) \; . \tag{9.6}$$

Hence, if (a) holds, Cartan integers are preserved. Moreover, we have

$$(\alpha, \beta) = \|\alpha\| \, \|\beta\| \cos\theta \; , \tag{9.7}$$

where $\theta$ is the angle between $\alpha$ and $\beta$. Hence,

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = 2 \frac{\|\alpha\|}{\|\alpha\|} \frac{\|\beta\|}{\|\alpha\|} \cos\theta = 2 \frac{\|\beta\|}{\|\alpha\|} \cos\theta \tag{9.8}$$

and

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \cos^2\theta \; . \tag{9.9}$$

Let $\theta$ be the angle between $\alpha$ and $\beta$ and let $\theta'$ be the angle between $f(\alpha)$ and $f(\beta)$. When the Cartan integers are preserved, then by (9.9) we have $4\cos^2\theta = 4\cos^2\theta'$, so $\cos^2\theta = \cos^2\theta'$, hence $\cos\theta = \pm\cos\theta'$. Now, (9.8) implies $\cos\theta = \cos\theta'$. Finally, since $0 \le \theta, \theta' \le \pi$, the equality $\cos\theta = \cos\theta'$ implies $\theta = \theta'$. Hence, angles between any two roots and ratios of lengths between any two non-orthogonal roots are preserved.

We thus need to show that (a) holds. Consider $\sigma' := f \circ \sigma_\alpha \circ f^{-1} \in \mathrm{GL}(E')$. This fixes $\Phi'$. Moreover, this fixes $f(P_\alpha)$ pointwise and maps $f(\alpha)$ to $-f(\alpha)$. Since $f$ is a vector space isomorphism, $f(P_\alpha)$ is a hyperplane in $E'$. Hence, $\sigma'$ is a reflection and so by Lemma 9.3 we have $\sigma' = \sigma_{f(\alpha)}$. ■

By the lemma the map sending $\sigma_\alpha$ to $f \circ \sigma_\alpha \circ f^{-1}$ induces an isomorphism between the Weyl groups of the root systems, so isomorphic root systems have isomorphic Weyl groups (hooray!). Notice that property (a) also says that the Weyl group $W$ is stable under conjugation in $\mathrm{Aut}(E, \Phi)$, i.e. it is a *normal* subgroup of $\mathrm{Aut}(E, \Phi)$.

We finish this subsection with a construction of new root systems. If we have a Lie algebra, then $\kappa$ induces an isomorphism between $H^*$ and $H$. We used this to define the element $t_\alpha \in H$ corresponding to a root $\alpha$, i.e. $\alpha = \kappa(t_\alpha, -)$. Recall from (8.4) the element $h_\alpha = 2\frac{t_\alpha}{\kappa(t_\alpha,t_\alpha)}$ that spans the semisimple part in the $\mathfrak{sl}_2$-triple for $\alpha$. Under the isomorphism, this corresponds to the element

$$\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)} \in H^* . \tag{9.10}$$

This is called the **dual** root of $\alpha$ (note that it is *not* a root of the Lie algebra). Of course we can define this also for any abstract root system. You can check that $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ forms another root system in $E$, called the **dual** root system. It is not necessarily isomorphic to $\Phi$ but the Weyl groups are isomorphic. Moreover, $\Phi^{\vee\vee}$ gives back $\Phi$, which is the reason for the word *dual*.

Review rank 2 cases; add $G_2$

**9.3. Examples.**

**9.4. Pairs of roots.** Recall the relations

$$\langle\alpha,\beta\rangle = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta \quad \text{and} \quad \langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\cos^2\theta \tag{9.11}$$

from (9.8) and (9.9). Both $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$ are integers of the same sign. Since $0 \le \cos^2\theta \le 1$, their product lies between 0 and 4. This only leaves a few possibilities for the Cartan integers, listed in Table III.1. Note that the (squared) length-ratio between roots is given by

$$\frac{\langle\alpha,\beta\rangle}{\langle\beta,\alpha\rangle} = \frac{(\alpha,\alpha)}{(\beta,\beta)} = \frac{\|\alpha\|^2}{\|\beta\|^2} . \tag{9.12}$$

Recall from (8.8) that for any two roots $\alpha,\beta$ ($\alpha \neq \pm\beta$) of a semisimple Lie algebra all vectors in the $\alpha$-string through $\beta$

$$\beta - r\alpha, \dots, \beta, \dots, \beta + q\alpha$$

are roots as well (here, $r$ resp. $q$ are chosen maximal such that the corresponding vector is a root). We proved this using the $\mathfrak{sl}_2$-representation theory. We can now prove this also abstractly for any root system using the following lemma.

***Lemma 9.5.*** *Let $\alpha,\beta \in \Phi$ with $\alpha \neq \pm\beta$.*

*(a) If $(\alpha,\beta) > 0$, then $\alpha - \beta \in \Phi$.*
*(b) If $(\alpha,\beta) < 0$, then $\alpha + \beta \in \Phi$.*

*Proof.* Assume that $(\alpha,\beta) > 0$. Then also $\langle\alpha,\beta\rangle > 0$. This implies that $\langle\alpha,\beta\rangle$ or $\langle\beta,\alpha\rangle$ equals 1 (see Table III.1). If $\langle\alpha,\beta\rangle = 1$, then $\Phi \ni \sigma_\beta(\alpha) = \alpha - \beta$ by (R3). Similarly, if $\langle\beta,\alpha\rangle = 1$, then $\beta - \alpha \in \Phi$, hence $\Phi \ni -(\beta-\alpha) = \alpha - \beta$. The case $(\alpha,\beta) < 0$ follows by applying the case just proven to $-\beta$ in place of $\beta$. ■

Now, consider in an (abstract) root system the $\alpha$-string through $\beta$

$$\beta - r\alpha, \dots, \beta, \dots, \beta + q\alpha$$

| $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$ | $\langle\alpha,\beta\rangle$ | $\langle\beta,\alpha\rangle$ | $\theta(\alpha,\beta)$ | $\|\alpha\|^2 / \|\beta\|^2$ |
|---|---|---|---|---|
| 0 | 0 | 0 | $\pi/2$ | undetermined |
| 1 | 1 | 1 | $\pi/3$ | 1 |
| 1 | $-1$ | $-1$ | $2\pi/3$ | 1 |
| 2 | 1 | 2 | $\pi/4$ | 1/2 |
| 2 | $-1$ | $-2$ | $3\pi/4$ | 1/2 |
| 2 | 2 | 1 | $\pi/4$ | 2 |
| 2 | $-2$ | $-1$ | $3\pi/4$ | 2 |
| 3 | 1 | 3 | $\pi/6$ | 1/3 |
| 3 | $-1$ | $-3$ | $5\pi/6$ | 1/3 |
| 3 | 3 | 1 | $\pi/6$ | 3 |
| 3 | $-3$ | $-1$ | $5\pi/6$ | 3 |
| 4 | 2 | 2 | 0 | 1 ($\alpha=\beta$) |
| 4 | $-2$ | $-2$ | $\pi$ | 1 ($\alpha=-\beta$) |

***Table III.1.*** Possibilities for Cartan integers. Note that $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 0 \Leftrightarrow \langle\alpha,\beta\rangle = 0 = \langle\beta,\alpha\rangle$, so this only leaves one case. Moreover, the other cases for the product being equal to 4 would imply that $\alpha = \pm 2\beta$ or $\alpha = \pm\frac{1}{2}\beta$, which is not possible.

with $r$ resp. $q$ chosen maximal such that the corresponding vector is a root. Suppose there is some $-r < i < q$ such that $\beta + i\alpha \notin \Phi$. Then one can find $p < s$ in this interval such that

$$\beta + p\alpha \in \Phi, \quad \beta + (p+1)\alpha \notin \Phi, \quad \beta + (s-1)\alpha \notin \Phi, \quad \beta + s\alpha \in \Phi\ .$$

But then Lemma 9.5 implies that

$$(\alpha,\alpha) + p(\alpha,\beta) = (\alpha,\beta+p\alpha) \geq 0\ , \quad (\alpha,\alpha) + s(\alpha,\beta) = (\alpha,\beta+s\alpha) \leq 0\ .$$

Since $(\alpha,\alpha) > 0$ and $p < s$, this is a contradiction. Hence, the string is "unbroken".

We can now even prove something more about root strings we didn't see so easily before in the Lie algebra context. In (8.9) we have seen that the root string length is given by the Cartan integer, i.e.

$$\langle\beta,\alpha\rangle = \beta(h_\alpha) = r - q\ .$$

From Table III.1 we thus conclude that *root strings are of length at most 4*.

## 10. Simple roots and Weyl group

We continue investigating an abstract root system $(E,\Phi)$ with Weyl group $W$.

**10.1. Bases and Weyl chambers.** By definition, the roots span $E$, so we can select a basis consisting of roots. Here's a definition for a particularly nice kind of basis.

***Definition 10.1***. A subset $\Delta$ of $\Phi$ is called a **base** if

(B1) $\Delta$ is a basis of $E$,

(B2) each root $\beta$ can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha$ with integral coefficients $k_\alpha$ all non-negative or non-positive.

After fixing a base, we call the roots in $\Delta$ **simple**. The **height** of a root $\beta$ is defined as the sum of the coefficients, i.e.

$$\operatorname{ht}\beta := \sum_{\alpha\in\Delta} k_\alpha \, . \tag{10.1}$$

If all $k_\alpha \geq 0$ (resp. all $k_\alpha \leq 0$), we call $\beta$ **positive** (resp. **negative**). The subset of positive (resp. negative) roots is denoted by $\Phi^+$ (resp. $\Phi^-$). Clearly, $\Phi^- = -\Phi^+$. We define a partial order $\prec$ on $E$ by

$$\mu \prec \lambda :\Leftrightarrow \lambda - \mu \text{ is a sum of positive roots} \, . \tag{10.2}$$

Note that $0 \prec \alpha$ for a root $\alpha$ if and only if $\alpha$ is positive. Angles between simple roots are *obtuse*:

***Lemma 10.2***. *If $\Delta$ is a base, then $(\alpha, \beta) \leq 0$ and $\alpha - \beta$ is not a root for any $\alpha \neq \beta$ in $\Delta$.*

*Proof.* Otherwise, $(\alpha, \beta) > 0$. Since $\alpha \neq \beta$ and $\Delta$ is a base, also $\alpha \neq -\beta$, so we can apply Lemma 9.5 which says that $\alpha - \beta$ is a root. But this violates the property (B2) of a base. ∎

So far, so great. The only problem is: we don't even know whether a base exists! We'll prove this now by explicitly constructing a base in a geometric way (the construction will in fact produce all possible bases). For a vector $\gamma \in E$ define

$$\Phi^+(\gamma) := \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\} \, . \tag{10.3}$$

This is the set of roots lying on the "positive" side of the hyperplane orthogonal to $\gamma$. We want to consider this for vectors not lying on any of the (finitely many) reflection hyperplanes $P_\alpha$, i.e. a for $\gamma \in E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$. Such vectors are called **regular**, otherwise **singular**. The following elementary lemma implies that regular vectors exist.

***Lemma 10.3***. *The union of finitely many hyperplanes in a finite-dimensional vector space over an* infinite *field is a proper subset of the vector space.*

*Proof.* Let $F$ be the ground field, let $F^n$ be the vector space and let $H_1, \ldots, H_N$ be the hyperplanes. Each $H_j$ is given by a linear equation $\sum_{i=1}^n a_{j,i} x_i = 0$ in the coordinates $x_i$ on $F^n$. For $t \in F$ consider the point in $F^n$ with coordinates $(t^i)_{i=1}^n$. If this point lies on $H_j$, it is a zero of the polynomial $p_j := \sum_{i=1}^n a_{j,i} T^i \in F[T]$. Since a polynomial has only finitely many zeros and $F$ is infinite, we can find $t \in F$ such that $p_j(t) \neq 0$ for all $j = 1, \ldots, n$. This means, the corresponding point $(t^i)_{i=1}^n$ does not lie on any $H_j$. ∎

The important property of a regular vector $\gamma$ is that any root lies either on the "positive" side or on the "negative" side of the hyperplane orthogonal to $\gamma$, i.e.

$$\Phi = \Phi^+(\gamma)\dot{\cup}-\Phi^+(\gamma) \; . \tag{10.4}$$

Call $\alpha \in \Phi^+(\gamma)$ **decomposable** if $\alpha$ can be written as a sum of roots in $\Phi^+(\gamma)$, **indecomposable** otherwise. Let $\Delta(\gamma)$ be the set of indecomposable roots.

***Theorem 10.4****. For any regular vector $\gamma$ the set $\Delta(\gamma)$ is a base, and every base is of this form.*

*Proof.* To prove (B2), it is by (10.4) sufficient to show that each root in $\Phi^+(\gamma)$ is a non-negative integral linear combination of roots in $\Delta(\gamma)$. Suppose this does not hold. Then some $\alpha \in \Phi^+(\gamma)$ cannot be written like that. Among those $\alpha$ choose one so that $(\gamma, \alpha)$ is minimal. Obviousy, $\alpha \notin \Delta(\gamma)$. By definition of $\Delta(\gamma)$, this means that $\alpha$ is decomposable, so $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma)$, whence $(\gamma, \alpha) = (\gamma, \beta_1)+(\gamma, \beta_2)$. But $(\gamma, \beta_i)$ is positive since $\beta_i \in \Phi^+(\gamma)$, so by minimality of the choice of $\alpha$, the $\beta_i$ must each be a non-negative integral linear combination of roots in $\Delta(\gamma)$. But then so is $\alpha = \beta_1 + \beta_2$, contradicting our assumption.

Since the roots span $E$, to prove that $\Delta(\gamma)$ is a basis (B1), it is now sufficient to prove that $\Delta(\gamma)$ is linearly independent. Suppose that $\sum_{\alpha\in\Delta(\gamma)} r_\alpha \alpha = 0$ for some $r_\alpha \in \mathbb{R}$. Separating the indices $\alpha$ for which $r_\alpha > 0$ from those for wich $r_\alpha < 0$, we can write this as $\sum_\alpha s_\alpha \alpha = \sum_\beta t_\beta \beta$ with disjoint indexing sets for the $\alpha, \beta$. Set $\epsilon \coloneqq \sum_\alpha s_\alpha \alpha = \sum_\beta t_\beta \beta$. Then $(\epsilon, \epsilon) = \sum_{\alpha,\beta} s_\alpha t_\beta (\alpha, \beta)$. We claim that each $(\alpha, \beta) \leq 0$, forcing $\epsilon = 0$. Otherwise, $(\alpha, \beta) > 0$ and therefore $\alpha - \beta$ is a root by Lemma 9.5. By the decomposition (10.4) either $\alpha - \beta \in \Phi^+(\gamma)$ or $\beta - \alpha \in \Phi^+(\gamma)$. In the first case, $\alpha = \beta + (\alpha - \beta)$ would imply that $\alpha$ is decomposable—a contradiction. In the second case, $\beta = \alpha + (\beta - \alpha)$ is decomposable—again a contradiction. Hence, $(\alpha, \beta) \leq 0$. Now, because of $\epsilon = 0$, we get $0 = (\gamma, \epsilon) = \sum_\alpha (\gamma, \alpha)$, forcing all $s_\alpha = 0$. Similarly, all $t_\beta = 0$. Hence, all $r_\alpha = 0$, proving that $\Delta(\gamma)$ is linearly independent.

It remains to prove the claim that all bases are of this form. So, let $\Delta$ be a base. For each $\alpha \in \Delta$ consider the positive half-space

$$H_\alpha^+ \coloneqq \{\gamma \in E \mid (\gamma, \alpha) > 0\} \; .$$

We claim that $\bigcap_{\alpha\in\Delta} H_\alpha^+$ is non-empty. This has just to do with the fact the $\Delta$ is a basis. Namely, for each $\alpha \in \Delta$ let $\alpha'$ be the projection of $\alpha$ on the orthogonal complement of the subspace spanned by all $\beta \in \Delta$ with $\beta \neq \alpha$. Let $\gamma \coloneqq \sum_{\alpha\in\Delta} \alpha'$. Then for each $\alpha$ we have $(\gamma, \alpha) = 1 > 0$. Hence, $\gamma$ lies in each $H_\alpha^+$. Now, take any such $\gamma$. Because of (B2) we then have $(\gamma, \alpha) > 0$ or $(\gamma, \alpha) < 0$ for all $\alpha \in \Phi$. In particular, $\gamma$ does not lie on any of the hyperplanes $P_\alpha$, so $\gamma$ is regular. Moreover, $\Phi^+ \subset \Phi^+(\gamma)$ and $-\Phi^+ \subset -\Phi^+(\gamma)$. Because $\Phi^+\dot{\cup} - \Phi^+ = \Phi = \Phi^+(\gamma)\dot{\cup} - \Phi^+(\gamma)$, we must have equality. So $\Phi^+ = \Phi^+(\gamma)$, and therefore $\Delta$ consists of indecomposable elements, i.e. $\Delta \subset \Delta(\gamma)$. Since both sets are a basis of $E$, we must have equality. ∎

***Remark 10.5***. In proving the linearly independence of $\Delta(\gamma)$ in the proof of Theorem 10.4 we have actually proven the following statement: a set of vectors in $E$ lying strictly on one side of a hyperplane in $E$ and forming pairwise obtuse angles must be linearly independent.

We want to analyze the effect of choosing different regular vectors in the construction of bases. Let's define a relation $\sim$ on the regular vectors in $E$ by $\gamma \sim \gamma'$ if and only if $\gamma$ and $\gamma'$ lie on the same side of each hyperplane $P_\alpha$ for all $\alpha \in \Phi$, i.e. $(\gamma, \alpha) > 0$ if and only if $(\gamma', \alpha) > 0$. This is clearly an equivalence relation, and so we get a partition of $E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$ into equivalence classes. The equivalence classes are called **Weyl chambers** of $E$. The Weyl chamber of a regular vector $\gamma$ is denoted by $\mathcal{C}(\gamma)$. We can describe this explicitly as follows. Define a function $s_\gamma \colon \Phi \to \{+,-\}$ according to the sign of $(\gamma, \alpha)$. Then clearly $\gamma \sim \gamma'$ if and only if $s_\gamma = s_{\gamma'}$, and the set of all such vectors $\gamma'$ is precisely

$$\mathcal{C}(\gamma) = \bigcap_{\alpha\in\Phi} H_\alpha^{s_\gamma(\alpha)} , \tag{10.5}$$

where

$$H_\alpha^+ \coloneqq \{\gamma' \in E \mid (\gamma', \alpha) > 0\} , \quad H_\alpha^- \coloneqq \{\gamma' \in E \mid (\gamma', \alpha) < 0\} . \tag{10.6}$$

As there are only finitely many functions $\Phi \to \{+,-\}$, it follows that there are only *finitely* many Weyl chambers. Moreover, they are open convex subsets of $E$. Note that not any choice of function $\Phi \to \{+,-\}$ needs to define a Weyl chamber as the corresponding intersection of half-spaces may be empty (look at the $A_2$ example); but if it is non-empty, it is a Weyl chamber. If you know topology: the Weyl chambers are precisely the connected components of $E \setminus \bigcup_{\alpha\in\Delta} P_\alpha$.

Why do we talk about Weyl chambers? In the construction of bases above we considered for a regular vector $\gamma$ the set $\Phi^+(\gamma)$ of all roots lying on the positive side of the hyperplane orthogonal to $\gamma$. Now we see that

$$\gamma \sim \gamma' \Leftrightarrow \Phi^+(\gamma) = \Phi^+(\gamma') \Leftrightarrow \Delta(\gamma) = \Delta(\gamma') . \tag{10.7}$$

Hence, *bases are in bijection with Weyl chambers*! We write $\mathcal{C}(\Delta)$ for the Weyl chamber defined by a base $\Delta$, i.e. $\mathcal{C}(\Delta) = \mathcal{C}(\gamma)$ when $\Delta = \Delta(\gamma)$. We call it the **fundamental Weyl chamber** relative to $\Delta$. Since we can decompose $\Phi = \Phi^+(\gamma)\dot\cup - \Phi^+(\Delta))$, we explicitly have

$$\mathcal{C}(\Delta) = \bigcap_{\alpha\in\Phi^+(\gamma)} H_\alpha^+ = \bigcap_{\alpha\in\Delta} H_\alpha^+ . \tag{10.8}$$

Since the Weyl group acts on the set of roots and acts by orthogonal transformations on $E$, it also acts on the Weyl chambers; explicitly,

$$\sigma\mathcal{C}(\gamma) = \mathcal{C}(\sigma\gamma) \tag{10.9}$$

for $\sigma \in W$. Similarly, $W$ acts on bases, sending $\Delta$ to $w(\Delta)$. These two actions are compatible with the correspondence between Weyl chambers and bases:

$$\sigma(\Delta(\gamma)) = \Delta(\sigma\gamma) . \tag{10.10}$$

**10.2. Lemmas on simple roots.** Fix a base $\Delta$ of $\Phi$. We prove here several useful lemmas about the behavior of simple roots.

***Lemma 10.6***. *If $\alpha$ is a positive root but not simple, then $\alpha - \beta$ is a positive root for some $\beta \in \Delta$.*

*Proof.* If $(\alpha, \beta) \leq 0$ for all $\beta \in \Delta$, then Remark 10.5 implies that $\Delta \cup \{\alpha\}$ is linearly independent—a contradiction since $\Delta$ is a basis. So $(\alpha, \beta) > 0$ for some $\beta \in \Delta$ and then $\alpha - \beta$ is a root by Lemma 9.5. Since $\alpha$ is a positive root, we can write $\alpha = \sum_{\gamma \in \Delta} k_\gamma \gamma$ with $k_\gamma \geq 0$. Moreover, since $\alpha \neq \pm\beta$, there is $\gamma \neq \beta$ with $k_\gamma > 0$. This coefficient $k_\gamma$ thus also appears in the representation of $\alpha - \beta$ as a linear combination of simple roots. Hence, there is a positive coefficient in this representation, and then all must be non-negative by (B2), i.e. $\alpha - \beta$ is positive. ■

***Corollary 10.7***. *Any positive root $\alpha$ can be written in the form $\alpha_1 + \ldots \alpha_k$ for $\alpha_i \in \Delta$ (not necessarily distinct) in such a way that each partial sum $\alpha_1 + \ldots + \alpha_i$ is a root.*

*Proof.* We prove this by induction on $\operatorname{ht} \alpha$ using Lemma 10.6. The claim is clear if $\operatorname{ht} \alpha = 1$, i.e. $\alpha$ is simple. Let $\operatorname{ht} \alpha > 1$. Then by Lemma 10.6 there is $\beta \in \Delta$ such that $\alpha - \beta$ is a positive root. We have $\alpha = (\alpha - \beta) + \beta$. Hence, $\operatorname{ht}(\alpha - \beta) < \operatorname{ht}(\alpha)$, so by induction $\alpha - \beta$ can be written as a sum of simple roots such that all partial sums are roots. The same then clearly holds for $(\alpha - \beta) + \beta = \alpha$ as well. ■

***Lemma 10.8***. *If $\alpha$ is a simple root, then $\sigma_\alpha$ permutes the positive roots other than $\alpha$.*

*Proof.* Let $\beta \in \Phi^+ \setminus \{\alpha\}$ and write $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ with $k_\gamma > 0$. It is clear that $\beta \neq \pm\alpha$, hence there is $\gamma \neq \alpha$ with $k_\gamma \neq 0$. Then the coefficient of $\gamma$ in $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ is still $k_\gamma$, so $\sigma_\alpha(\beta)$ has at least one positive coefficient, forcing $\sigma_\alpha(\beta)$ to be positive by (B2). Moreover, since $\beta \neq \pm\alpha$, also $\sigma_\alpha(\beta) \neq \alpha$. ■

Consider the element

$$\delta := \frac{1}{2} \sum_{\alpha \succ 0} \alpha \,, \tag{10.11}$$

i.e. the "half-sum of positive roots". This element will be useful later as the Weyl group acts on it in a special way. Namely, Lemma 10.8 immediately implies:

***Corollary 10.9***. *For any simple root $\alpha$ we have $\sigma_\alpha(\delta) = \delta - \alpha$.* ■

***Lemma 10.10***. *Let $\alpha_1, \ldots, \alpha_t \in \Delta$ (not necessarily distinct). Write $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \cdots \sigma_{t-1}(\alpha_t)$ is negative, there is some index $1 \leq s < t$ such that*

$$\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1} \sigma_{s+1} \cdots \sigma_{t-1} \,,$$

*i.e. $\sigma_s$ and $\sigma_t$ can be "deleted" in the expression $\sigma_1 \cdots \sigma_t$.*

*Proof.* For $0 \le i < t$ set $\beta_i \coloneqq \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t)$. Note that for $i = t-1$ we have the empty product, so $\beta_{t-1} \coloneqq \alpha_t$. By assumption, $\beta_0$ is negative. Since $\beta_{t-1}$ is positive, we can find a smallest index $s$ such that $\beta_s$ is positive. Then $\sigma_s(\beta_s) = \beta_{s-1}$ is negative, so $\beta_s = \alpha_s$ by Lemma 10.8. Let $\sigma \coloneqq \sigma_{s+1} \cdots \sigma_{t-1}$. Then $\sigma(\alpha_t) = \beta_s = \alpha_s$. Recall that $\sigma_{\sigma(\alpha_t)} = \sigma\sigma_{\alpha_t}\sigma^{-1}$ by Lemma 9.4. Hence,

$$\sigma_s = \sigma_{\sigma(\alpha_t)} = \sigma\sigma_{\alpha_t}\sigma^{-1} = \sigma_{s+1} \cdots \sigma_{t-1}\sigma_t\sigma_{t-1} \cdots \sigma_{s+1} \ ,$$

so

$$\begin{aligned} \sigma_1 \cdots \sigma_t &= \sigma_1 \cdots \sigma_{s-1}\sigma_s\sigma_{s+1} \cdots \sigma_t \\ &= \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}\sigma_t\sigma_{t-1} \cdots \sigma_{s+1}\sigma_{s+1} \cdots \sigma_t \\ &= \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1} \ , \end{aligned}$$

where we used the fact that reflections are of order 2. ■

***Corollary 10.11***. *If $\sigma = \sigma_1 \cdots \sigma_t$ is an expression of $\sigma \in W$ in terms of reflections in simple roots with $t$ minimal, then $\sigma(\alpha_t)$ is a* negative *root.*

*Proof.* Otherwise, we could delete two factors by Lemma 10.10, contradicting the minimality of the expression. ■

**10.3. The Weyl group.** We prove that the Weyl group $W$ acts *simply transitively* on the Weyl chambers (and thus on the bases). Moreover, for any fixed choice of base $\Delta$, the group $W$ is generated by the **simple reflections** $\sigma_\alpha$, $\alpha \in \Delta$, relative to this base.

***Theorem 10.12***. *Let $\Delta$ be a base.*

*(a) If $\gamma$ is a regular vector, then there is $\sigma \in W$ such that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$.*
*(b) $W$ acts transitively on the Weyl chambers, i.e. if $\mathcal{C}, \mathcal{C}'$ are Weyl chambers, there is $\sigma \in W$ such that $\sigma\mathcal{C}' = \mathcal{C}$.*
*(c) If $\alpha$ is any root, there exists $\sigma \in W$ such that $\sigma(\alpha) \in \Delta$.*
*(d) $W$ is generated by the simple reflections $\sigma_\alpha$, $\alpha \in \Delta$.*
*(e) $W$ acts simply on the Weyl chambers, i.e. if $\sigma\mathcal{C} = \mathcal{C}$, then $\sigma = 1$.*

*Proof.* Let $W'$ be the subgroup of $W$ generated by $\sigma_\alpha$ for $\alpha \in \Delta$. We prove the statements first for $W'$, then deduce that $W = W'$.

(a) Let $\delta = \frac{1}{2}\sum_{\alpha \succ 0} \alpha$ and choose $\sigma \in W'$ such that $(\sigma(\gamma), \delta)$ is maximal. If $\alpha \in \Delta$, then $\sigma_\alpha\sigma \in W'$, so by the choice of $\sigma$ we have

$$(\sigma(\gamma), \delta) \ge (\sigma_\alpha\sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta - \alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha) \ ,$$

where we used Corollary 10.9. This forces $(\sigma(\gamma), \alpha) \ge 0$. Since $\gamma$ is regular, it is not orthogonal to any root, so we cannot have $0 = (\gamma, \sigma^{-1}\alpha) = (\sigma(\gamma), \alpha)$. Hence, $(\sigma(\gamma), \alpha) > 0$ for any $\alpha \in \Delta$.

(b) Let $\mathcal{C}'$ be any Weyl chamber and choose some $\gamma \in \mathcal{C}'$. We have just proven that there is $\sigma \in W'$ with $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$, i.e.

$$\sigma(\gamma) \in \bigcap_{\alpha \in \Delta} H_\alpha^+ = \mathcal{C}(\Delta) ,$$

i.e. $\sigma\mathcal{C}' = \mathcal{C}(\Delta)$. It follows that $W'$ acts transitively on the Weyl chambers.

(c) Since we proved that $W'$ acts transitively on the Weyl chambers, and thus on the bases, it is enough to prove that any root $\alpha$ is contained in some base. For $\beta \in \Phi \setminus \{\pm\alpha\}$ the hyperplanes $P_\alpha$ and $P_\beta$ are distinct. Therefore, $P_\alpha + P_\beta$ is the whole space $E$, so

$$\begin{aligned}\dim(P_\alpha \cap P_\beta) &= \dim P_\alpha + \dim P_\beta - \dim(P_\alpha + P_\beta) \\ &= \dim E - 1 + \dim E - 1 - \dim E \\ &= \dim E - 1 = \dim P_\alpha - 1 .\end{aligned}$$

Hence, $P_\alpha \cap P_\beta$ is a hyperplane in $P_\alpha$. By Lemma 10.3 the complement

$$P_\alpha \setminus \bigcup_{\beta \neq \pm\alpha} P_\beta$$

is non-empty, so we can find $\gamma \in P_\alpha$ with $\gamma \notin P_\beta$ for all $\beta \neq \pm\alpha$. This means $(\gamma, \alpha) = 0$ and $(\gamma, \beta) \neq 0$ for all $\beta \neq \pm\alpha$. Going a tiny bit away from the hyperplane $P_\alpha$, we can find $\gamma'$ close enough to $\gamma$ so that $(\gamma', \alpha) = \epsilon > 0$ and $|(\gamma', \beta)| > \epsilon$ for all $\beta \neq \pm\alpha$. Clearly, $\alpha \in \Phi^+(\gamma')$. Also, $\alpha$ is decomposable: if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma')$, then

$$\epsilon = (\gamma', \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2) > \epsilon .$$

(d) It is enough to show that each $\sigma_\alpha$ for $\alpha \in \Phi$ is contained in $W'$. By (the proof of) (c) there is $\sigma \in W'$ such that $\beta := \sigma(\alpha) \in \Delta$. Then $\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$ by Lemma 9.4, so $\sigma_\alpha = \sigma^{-1}\sigma_\beta\sigma \in W'$.

(e) Suppose $\sigma(\Delta) = \Delta$ but $\sigma \neq 1$. We can now write $\sigma$ minimally as a product $\sigma_1 \cdots \sigma_t$ of simple reflections $\sigma_i = \sigma_{\alpha_i}$ for $\alpha_i \in \Delta$. Then $\sigma(\alpha_t)$ is a negative root by Corollary 10.11. But $\sigma(\alpha_t) \in \sigma(\Delta) = \Delta$ is positive—a contradiction.

■

Now that we have a special generating set $\{\sigma_\alpha \mid \alpha \in \Delta\}$ for $W$ we want to investigate this a bit more closely. First, among all the ways an element $\sigma \in W$ can be written as a product $\sigma_1 \cdots \sigma_t$ it makes sense to consider those of *minimal* length $t$. Such an expression is called a **reduced expression** of $\sigma$ and its length is called the **length** of $\sigma$, denoted $\ell(\sigma)$. We can characterize the length geometrically.

***Lemma 10.13***. *The length of an element $\sigma \in W$ is equal to the number of positive roots $\alpha$ for which $\sigma(\alpha)$ is negative.*

*Proof.* Let $n(\sigma)$ be the number of positive roots mapped to negative roots. We prove the claim by induction on $\ell(\sigma)$. The case $\ell(\sigma) = 0$ is clear: this implies $\sigma = 1$, so $n(\sigma) = 0$. Assume now that $\ell(\sigma) > 0$. Write $\sigma$ in reduced form as $\sigma = \sigma_1 \cdots \sigma_t$, where $\sigma_i = \sigma_{\alpha_i}$. Set $\alpha \coloneqq \alpha_t$. From Corollary 10.11 we know that $\sigma(\alpha)$ is negative. Recall from Lemma 10.8 that $\sigma_\alpha$ permutes the positive roots other than $\alpha$. Hence, $n(\sigma\sigma_\alpha) = n(\sigma) - 1$. On the other hand, $\ell(\sigma\sigma_\alpha) = \ell(\sigma) - 1 < \ell(\sigma)$, so by induction $\ell(\sigma\sigma_\alpha) = n(\sigma\sigma_\alpha)$. Combining the statements yields $\ell(\sigma) = n(\sigma)$. ∎

We denote by $\overline{\mathcal{C}(\Delta)}$ the (topological) closure in $E$ of the fundamental Weyl chamber $\mathcal{C}(\Delta)$ relative to $\Delta$. This simply means we add to $\mathcal{C}(\Delta)$ the segments of its bounding hyperplanes.

***Lemma 10.14.*** *The set $\overline{\mathcal{C}(\Delta)}$ is a **fundamental domain** for the action of $W$ on $E$, i.e. each vector in $E$ is in the $W$-orbit of precisely one point of $\overline{\mathcal{C}(\Delta)}$.*

*Proof.* We first show that if $\lambda, \mu \in \overline{\mathcal{C}(\Delta)}$ with $\sigma\lambda = \mu$, then already $\lambda = \mu$. We prove this by induction on $\ell(\sigma)$. The case $\ell(\sigma) = 0$ is clear, so assume $\ell(\sigma) > 0$. Then $\sigma$ must send some positive root to a negative root by Lemma 10.13. So, $\sigma$ cannot send all simple roots to positive roots (since then all positive roots would be send to positive roots). Let $\alpha \in \Delta$ be such that $\sigma(\alpha)$ is negative. Since $\mu, \lambda \in \overline{\mathcal{C}(\Delta)}$, we then have

$$0 \geq (\mu, \sigma\alpha) = (\sigma^{-1}\mu, \alpha) = (\lambda, \alpha) \geq 0 \,.$$

This forces $(\lambda, \alpha) = 0$, so $\lambda \in P_\alpha$, hence $\sigma_\alpha(\lambda) = \lambda$ and therefore $\sigma\sigma_\alpha(\lambda) = \sigma(\lambda) = \mu$. Since $\sigma(\alpha)$ is negative, it follows from Lemma 10.13 that $\ell(\sigma\sigma_\alpha) = \ell(\sigma) - 1 < \ell(\sigma)$. Hence, by induction we conclude that $\lambda = \mu$. ∎

Prove second part, i.e. that each vector lies in an orbit.

**10.4. Irreducible root systems.** We call a subset of $E$ **irreducible** if it cannot be partitioned into the union of two proper subsets such that each vector in one set is orthogonal to each vector in the other. We say that a root system is **irreducible** if the set $\Phi$ of roots is irreducible.

***Lemma 10.15.*** *$\Phi$ is irreducible if and only if one (any) base $\Delta$ is irreducible.*

*Proof.* Suppose that $\Phi$ is not irreducible, i.e. $\Phi = \Phi_1 \cup \Phi_2$ and $(\Phi_1, \Phi_2) = 0$. If $\Delta$ would be contained in $\Phi_1$, then $(\Delta, \Phi_2) = 0$, so $(E, \Phi_2) = 0$ since $\Delta$ is a basis of $E$. This is a contradiction. Similarly, $\Delta$ cannot be contained in $\Phi_2$. Hence, by taking from each $\Phi_1, \Phi_2$ the vectors from $\Delta$ we get a an orthogonal decomposition of $\Delta$.

Conversely, let $\Phi$ be irreducible but assume that $\Delta = \Delta_1 \cup \Delta_2$ with $(\Delta_1, \Delta_2) = 0$. From Theorem 10.12 we know that each root is in the $W$-orbit of a simple root. Hence, by letting $\Phi_i$ be the set of roots lying in the $W$-orbit of $\Delta_i$, we get a decomposition $\Phi = \Phi_1 \cup \Phi_2$. From the formula of a reflection you can see that each root in $\Phi_i$ is obtained from one in $\Delta_i$ by adding or subtracting roots in $\Delta_i$. Hence, $\Phi_i$ lies in the subspace $E_i$ spanned by $\Delta_i$, so $(\Phi_1, \Phi_2) = 0$. But this forces $\Phi_1 = \emptyset$ or $\Phi_2 = \emptyset$ since $\Phi$ is irreducible. Whence, $\Delta_1 = \emptyset$ or $\Delta_2 = \emptyset$, i.e. $\Delta$ is irreducible. ∎

Can be more precise

***Lemma 10.16***. *If $\Phi$ is irreducible, then with respect to the partial order $\prec$ there is a unique maximal root. If $\beta$ is this root, then $(\beta, \alpha) \geq 0$ for all $\alpha \in \Delta$, so $\beta$ is contained in the fundamental chamber $\overline{\mathcal{C}(\Delta)}$, and if $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ is the basis representation, then $k_\alpha > 0$ for all $\alpha \in \Delta$.*

*Proof.* Before proving uniqueness, we first prove the claimed properties of a maximal root. So, let $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ be maximal root. This must be a positive root since if $\beta$ were negative, $-\beta$ would be positive and larger than $\beta$. Let $\Delta_1 = \{\alpha \in \Delta \mid k_\alpha > 0\}$ and $\Delta_2 = \{\alpha \in \Delta \mid k_\alpha = 0\}$. Then $\Delta = \Delta_1 \cup \Delta_2$. Suppose $\Delta_2 \neq \emptyset$. Since distinct vectors in a base are obtuse by Lemma 10.2, we have $(\alpha, \beta) \leq 0$ for any $\alpha \in \Delta_2$ (apply the lemma to the simple roots in the basis expression of $\beta$). Since $\Phi$, and thus $\Delta$, is irreducible, there must be some $\alpha \in \Delta_2$ and some $\alpha' \in \Delta_1$ which are non-orthogonal, so $(\alpha, \alpha') < 0$ since base vectors are always obtuse by Lemma 10.2. Hence, $(\alpha, \beta) < 0$. Lemma 9.5 now implies that $\alpha + \beta$ is a root, contradicting the maximality of $\beta$. Therefore, $\Delta_2 = \emptyset$, i.e. $k_\alpha > 0$ for all $\alpha \in \Delta$. The above argument also shows that $(\alpha, \beta) \geq 0$ for all $\alpha \in \Delta$. Moreover, $(\alpha, \beta) > 0$ for some $\alpha \in \Delta$ since $\Delta$ spans $E$.

Now, we prove uniqueness. Let $\beta'$ be another maximal root. The preceding argument applies to $\beta'$ as well, so $(\alpha, \beta') \geq 0$ for all $\alpha \in \Delta$ and $(\alpha, \beta') > 0$ for some $\alpha \in \Delta$. Hence, $(\beta, \beta') > 0$. If $\beta \neq \beta'$, then Lemma 9.5 implies that $\beta - \beta'$ is a root. But then either $\beta \prec \beta'$ or $\beta' \prec \beta$, contradicting maximality. Hence, $\beta = \beta'$. ∎

***Lemma 10.17***. *Let $\Phi$ be irreducible. Then $W$ acts irreducibly on $E$. In particular, the $W$-orbit of a root $\alpha$ spans $E$.*

*Proof.* Let $E'$ be a non-zero $W$-invariant subspace of $E$. Since $W$ acts by orthogonal transformations, the orthogonal complement $E''$ of $E'$ is $W$-invariant as well, and $E = E' \oplus E''$. We claim that each root $\alpha \in \Phi$ lies either in $E'$ or in $E''$. Suppose $\alpha \notin E'$. If $\beta \in E'$, then, since $\sigma_\alpha(E') = E'$, we have $E' \ni \sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$, so $\langle \beta, \alpha \rangle \alpha \in E'$, hence $\langle \beta, \alpha \rangle = 0$, i.e. $(\beta, \alpha) = 0$. This shows that $E' \subseteq P_\alpha$, hence the orthogonal complement $E''$ of $E'$ contains the orhtogonal complement of $P_\alpha$, which is $\langle \alpha \rangle$. In particular, $\alpha \in E''$. We thus get a partition of $\Phi$ into orthogonal subsets, forcing one or the other to be empty since $\Phi$ is irreducible. Since $\Phi$ spans $E$, we conclude that $E' = E$. This shows that $W$ acts irrducibly.

The span of the $W$-orbit of a root is a non-zero $W$-invariant subspace of $E$, hence all of $E$ since $W$ acts irreducibly. ∎

***Lemma 10.18***. *Let $\Phi$ be irreducible. Then at most* two *root lengths occur in $\Phi$, and all roots of a given length lie in a single $W$-orbit.*

*Proof.* If $\alpha, \beta$ are arbitrary roots, not all $\sigma(\alpha)$, $\sigma \in W$, can be orthogonal to $\beta$ since the $\sigma(\alpha)$ span $E$ by Lemma 10.17. If $(\alpha, \beta) \neq 0$, then $\langle \alpha, \beta \rangle \neq 0$, and we know from Table III.1 that the possible ratios of squared root lengths of $\alpha$ and $\beta$ are 1, 2, 3, 1/2, 1/3. A presence of a third root length would yield also a ratio 3/2, so this cannot happen.

Let $\alpha, \beta$ be two distinct roots of equal length. As above, not all $\sigma(\alpha)$ can be orthogonal to $\beta$, so we can replace $\alpha$ by a root non-orthogonal to $\beta$. By Table III.1, ths forces $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1$. If this is $-1$, we can replace $\beta$ by $-\beta$ (which clearly lies in the same orbit, as it is the image under $\sigma_\beta$), so we can assume that $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = 1$. Then

$$(\sigma_\alpha \sigma_\beta \sigma_\alpha)(\beta) = \sigma_\alpha \sigma_\beta(\beta - \alpha) = \sigma_\alpha(-\beta - \alpha + \beta) = \alpha \ . \qquad \blacksquare$$

If $\Phi$ is irreducible with two root lengths, we can speak of **long roots** and **short roots**. If all roots have the same length, it is conventional to call all of them long.

***Lemma 10.19***. *Let $\Phi$ be irreducible with two distinct root lengths. Then the maximal root of $\beta$ is long.*

*Proof.* Let $\beta$ be the maximal root (Lemma 10.16). We need to show that $(\beta, \beta) \geq (\alpha, \alpha)$ for all $\alpha \in \Phi$. By Lemma 10.14 we can replace $\alpha$ by a root in the $W$-orbit of $\alpha$ lying in the fundamental chamber $\overline{\mathcal{C}(\Delta)}$. Note that also $\beta \in \overline{\mathcal{C}(\Delta)}$ by Lemma 10.16. Since $\beta$ is maximal, we have $\beta \succ \alpha$, so $\beta - \alpha \succ 0$, i.e. $\beta - \alpha$ is positive. Hence, $(\gamma, \beta - \alpha) \geq 0$ for all $\gamma \in \overline{\mathcal{C}(\Delta)}$. Applying this to $\gamma = \beta$ and $\gamma = \alpha$ yields

$$(\beta, \beta) \geq (\beta, \alpha) \geq (\alpha, \alpha) \ . \qquad \blacksquare$$

## 11. Classification

We will derive a combinatorial classification of all root systems up to isomorphism. This is a key step in the classification of semisimple Lie algebras. Let $(E, \Phi)$ be a root system, let $\ell$ be its rank, let $W$ be its Weyl group, and let $\Delta$ be a base.

### 11.1. Cartan matrix of $\Phi$.

Remember that an isomorphism of root systems preserves the Cartan integers (by definition). So, the Cartan integers are some discrete invariants of a root system and we may ask to what extend they determine the root system. We will see shortly: they already determine it completely!

Let's look at the Cartan integers on a base. Fix an ordering $(\alpha_1, \ldots, \alpha_\ell)$ of the simple roots. The matrix $C_\Phi \coloneqq (\langle \alpha_i, \alpha_j \rangle)$ is called the **Cartan matrix** of $\Phi$. The matrix of course depends on the chosen ordering, but this is not very serious as it just leads to a permutation of the indices in the matrix. We will thus always consider Cartan matrices up to *index permutations*. The important point is that the Cartan matrix is independent (up to index permutations) of the choice of $\Delta$, thanks to the fact that $W$ acts transitively on the bases by Theorem 10.12. Note that the diagonal entries of the Cartan matrix are all equal to 2 and that the off-diagonal entries are all negative by Lemma 10.2 (simple roots are obtuse). Moreover, the Cartan matrix is non-singular, which can be seen with the same argument as in Section 8.5 in the context of Lie algebras. Here's the promised feature.

***Proposition 11.1.*** *Let $(E', \Phi')$ be another root system and let $\Delta' = \{\alpha'_1, \dots, \alpha'_\ell\}$ be a base. If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for $1 \le i, j \le \ell$, then the map $\alpha_i \mapsto \alpha'_i$ extends (uniquely) to an isomorphism $f \colon (E, \Phi) \to (E', \Phi')$ of root systems. In particular, the Cartan matrix completely determines a root system up to isomorphism.*

*Proof.* Since $\Delta$ (resp. $\Delta'$) is a basis of $E$ (resp. $E'$) there is a unique vector space map $f \colon E \to E'$ mapping $\alpha_i \to \alpha'_i$ for all $1 \le i \le \ell$, and of course this is an isomorphism of vector spaces. To prove that $f$ is an isomorphism of root systems, it is by Lemma 9.4 sufficient to show that $f(\Phi) = \Phi'$; so far, we only know this for the *simple* roots. Since $f$ preserves by assumption the Cartan integers on the base, we have for simple roots $\alpha, \beta \in \Delta$ the relation

$$\begin{aligned} f(\sigma_\alpha(\beta)) &= f(\beta - \langle \beta, \alpha \rangle \alpha) = f(\beta) - \langle \beta, \alpha \rangle f(\alpha) \\ &= f(\beta) - \langle f(\beta), f(\alpha) \rangle f(\alpha) = \sigma_{f(\alpha)}(f(\beta)) \ . \end{aligned}$$

Since $\Delta$ is a basis, this equality holds for all vectors $\beta \in E$, so

$$\sigma_{f(\alpha)} = f \circ \sigma_\alpha \circ f^{-1}$$

for $\alpha \in \Delta$. Since the respective Weyl groups $W$ and $W'$ are generated by the simple reflections (Theorem 10.12), it follows that the map $\sigma \mapsto f \circ \sigma \circ f^{-1}$ is a group isomorphism from $W$ to $W'$. In particular, $f \circ \sigma \circ f^{-1}$ preserves the roots $\Phi'$ for any $\sigma \in W$. Now, by 10.12, each root $\beta \in \Phi$ lies in the $W$-orbit of some simple root $\alpha \in \Delta$, say $\beta = \sigma(\alpha)$. Since $\alpha$ is simple, we know that $f(\alpha)$ is a (simple) root. Hence,

$$f \circ \sigma \circ f^{-1}(f(\alpha)) = f \circ \sigma(\alpha) = f(\beta)$$

is a root. This proves that $f$ maps $\Phi$ to $\Phi'$, hence $f$ is an isomorphism of root systems. ■

**11.2. Coxeter graphs and Dynkin diagrams.** As a further helpful combinatorial structure we associate to a Cartan matrix a graph in the following way. As before, fix an ordering $(\alpha_1, \dots, \alpha_\ell)$ of the simple roots. The **Coxeter graph** of $\Phi$ is the graph having $\ell$ vertices and the $i$-th vertex is joined to the $j$-th ($i \neq j$) by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges (undirected). A look at Table III.1 shows that the number of edges is between 0 and 3. The reason this information is interesting is that the product $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ is precisely the order of the element $\sigma_{\alpha_i} \sigma_{\alpha_j}$ in the Weyl group, and these orders completely determine the Weyl group. Note that an index permutation on the Cartan matrix (the only amount of freedom) just amounts to a relabeling of the vertices of the Coxeter graph.

Add a proof of this.

But can we recover the Cartan matrix from the Coxeter graph? When we look at Table III.1 we see that for a given product $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ (i.e. for the number of edges between $\alpha_i$ and $\alpha_j$) there are always at most two possibilities for the Cartan integers (recall that the Cartan integers are negative since the roots are all simple, thus obtuse). More precisely, only if the two roots have unequal length, there are two possible cases; otherwise there's only one possibility. To nail down the case, we enhance the Coxeter graph as follows: if $\alpha_i$ and $\alpha_j$ have unequal length, we draw a common arrow over the edges pointing to the shorter of the two roots. It is then clear which of the two possible cases holds, and we can completely recover the Cartan matrix. The resulting diagram is called the **Dynkin diagram** of $\Phi$. Since the Cartan matrix completely determines the root system, so does the Dynkin diagram. As before, an index permutation on the Cartan matrix just amounts to a relabeling of the vertices of the Dynkin diagram.

**11.3. Irreducible components.** It is clear that $\Phi$ is irreducible if and only if the Coxeter graph is *connected*. In general, connected components of the Coxeter graph correspond to decompositions of $\Delta$ into mutually orthogonal subsets. Let $\Delta = \Delta_1 \cup \ldots \cup \Delta_t$ be such a decomposition. Let $E_i$ be the span of $\Delta_i$. Then it is clear that $E = E_1 \oplus \ldots \oplus E_t$, and this sum is orthogonal. Moreover, the set of roots $\Phi_i$ which are integral linear combinations of roots in $\Delta_i$ form a root system in $E_i$ whose Weyl group $W_i$ is the restriction to $E_i$ of the subgroup of $W$ generated by $\sigma_\alpha$ for $\alpha \in \Delta_i$. The Weyl group $W$ is isomorphic to the product $W_1 \times \ldots \times W_t$. The roots $\Phi$ decompose as $\Phi = \Phi_1 \cup \ldots \cup \Phi_t$, the components being mutually orthogonal. Also, the Cartan matrix has block diagonal form, with the blocks being the Cartan matrices of the $\Phi_i$.

***Proposition 11.2.*** $(E, \Phi)$ *decomposes uniquely into a direct sum of irreducible root systems* $(E_i, \Phi_i)$. ∎

**11.4. Classification theorem.** The discussion in the last section shows that to classify all root systems, it is sufficient to classify the *irreducible* root systems, or equivalently, the *connected* Dynkin diagrams.

***Theorem 11.3.*** *If $\Phi$ is an irreducible root system of rank $\ell$, its Dynkin diagram is one of the following ($\ell$ vertices in each case):*

Add diagrams and add proof. Proof in Hum is ok.

## 12. Construction of root systems and automorphisms

In the last section, we have described all the Dynkin diagrams that can possibly arise from a root system. Now, we prove that for each such diagram there is in fact a root system with this Dynkin diagram. This shows that isomorphism classes of root systems are in 1:1 correspondence with Dynkin diagrams (up to relabeling the vertices).

**12.1. Construction of types $A$–$G$.**

***Theorem 12.1.*** *For each Dynkin diagram there exists a root system having the given diagram.*

Add proof. I would actually take construction from classical Lie algebras (need to discuss this in §8), and use computer to verify for the exceptionals.

**12.2. Automorphisms.** We are going to give a complete description of the automorphism group of a root system $\Phi$. Recall that $W$ is a normal subgroup of

Aut $\Phi$. Fix a base $\Delta$ and consider the set

$$\Gamma := \{\sigma \in \operatorname{Aut}\Phi \mid \sigma(\Delta) = \Delta\} \tag{12.1}$$

of all automorphisms fixing this base. Evidently, this is a subgroup of Aut $\Phi$. If $\tau \in \Gamma \cap W$, then $\tau = 1$ by virtue of the action of $W$ on the bases being simple (Theorem 10.12). Moreover, if $\tau \in \operatorname{Aut}\Phi$, then $\tau(\Delta)$ is evidently another base, so, by transitivity of the action of $W$ on bases (Theorem 10.12), there exists $\sigma \in W$ such that $\sigma\tau(\Delta) = \Delta$, whence $\tau \in \Gamma W$. We have thus shown that

$$\Gamma \cap W = \{1\} \quad \text{and} \quad \operatorname{Aut}\Phi = \Gamma W \ , \tag{12.2}$$

in other words, Aut $\Phi$ is the *semidirect product* of $W$ and $\Gamma$:

$$\operatorname{Aut}\Phi = W \rtimes \Gamma \ . \tag{12.3}$$

There's a nice description of the automorphisms in $\Gamma$. First, by definition, any $\tau \in \Gamma$ defines a permutation on the set $\Delta$. Moreover, since $\tau$ is an automorphism of the root system, the Cartan integers, and thus their products, are preserved. Finally, $\tau$ preserves the length ratios of non-orthogonal roots by Lemma 9.4. Hence, $\tau$ is a **diagram automorphism** of the Dynkin diagram, i.e. a permutation on the vertices preserving the number of edges between vertices and their common direction if they are directed.

Give examples and list tables

## 13. Abstract theory of weights

Will only become important later in the rep theory; skipped for now.

# CHAPTER IV

# ISOMORPHISM AND CONJUGACY THEOREMS

## 14. Isomorphism theorem

From the abstract theory of root systems we go back to the concrete setting of Lie algebras. Our aim in this section is to prove that two semisimple Lie algebras having isomorphic root system are isomorphic. Hence, semisimple Lie algebras are completely determined by their root systems. What remains to be done afterwards (in the next chapter) is to show that for each root systems there actually *is* a semisimple Lie algebra with this root system. We know this already for the classical types $A$ to $D$; but we don't know this yet for types $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$.

Recall that for a semisimple Lie algebra $L$ over a field $F$ of characteristic 0 we choose a maximal toral subalgebra $H$ of $L$ and denote by $\Phi \subset H^*$ the set of roots of $L$ relative to $H$. The real span of the roots in $H^*$ gives a vector space $E$. This space is equipped with an inner product, coming from the restriction of the Killing to $H$.

### 14.1. Reduction to the simple case.

***Proposition 14.1***. *Let $L$ be a simple Lie algebra. Then the root system $\Phi$ of $L$ is an* irreducible *root system.*

*Proof.* Suppose $\Phi$ is not irreducible. Then we can decompose $\Phi = \Phi_1 \cup \Phi_2$ with two non-empty orthogonal subsets $\Phi_i$. If $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, then

$$(\alpha + \beta, \alpha) = (\alpha, \alpha) + (\beta, \alpha) = (\alpha, \alpha) \neq 0$$

and similarly $(\alpha+\beta, \beta) \neq 0$. If $\alpha+\beta$ were a root, it would be contained in either $\Phi_1$ or in $\Phi_2$, so we would have $(\alpha + \beta, \alpha) = 0$ or $(\alpha + \beta, \beta) = 0$. This not being the case, we conclude that $\alpha + \beta$ is not a root. From Lemma 8.3 we then see that $[L_\alpha L_\beta] \subset L_{\alpha+\beta} = 0$. Let $K$ be the subalgebra of $L$ generated by the $L_\alpha$ for $\alpha \in \Phi_1$. What we just said implies that all $L_\beta$, $\beta \in \Phi_2$, are contained in the centralizer $C_L(K)$ of $K$ in $L$. If $K$ were all of $L$, then $C_L(K) = Z(L) = 0$ because $L$ is simple—a contradiction because $K \neq 0$. Hence, $K$ is a proper subalgebra of $L$. If $\alpha, \beta \in \Phi_1$ is such that $\alpha + \beta$ is a root, it must be contained in $\Phi_1$. This shows that $[L_\alpha K] \subseteq K$ for all $\alpha \in \Phi_1$, i.e. $K$ is normalized by all $L_\alpha$, $\alpha \in \Phi_1$. Moreover, as argued above, $[L_\beta K] = 0$ for $\beta \in \Phi_2$, so $K$ is normalized by all $L_\alpha$,

$\alpha \in \Phi$, hence by all of $L$. This means that $K$ is actually an ideal in $L$. As $K$ is non-trivial, this contradicts the simplicity of $L$. ■

Next, let $L$ be an arbitrary semisimple Lie algebra. Recall that $L$ can be written uniquely as a direct sum $L = L_1 \oplus \ldots \oplus L_t$ of simple ideals $L_t$. We are going to show that this decomposition of $L$ is in accordance with the decomposition of the root system $\Phi$ into irreducible components. This relies on the following general lemma ([1, Exercise 5.8]).

***Lemma 14.2.*** *Let $L = L_1 \oplus \ldots \oplus L_t$ be the decomposition into simple ideals. Then the semisimple and nilpotent parts of $x \in L$ are the sums of the semisimple and nilpotent parts in the various $L_i$ of the components of $x$.*

*Proof.* Let $x = \sum_{i=1}^t x_i$ with $x_i \in L$. For each $i$ let $x_i = x_{i,s} + x_{i,n}$ be the decomposition of $x_i$ into semisimple and nilpotent components in $L_i$. The restriction $\mathrm{ad}_L\, x_{i,s}|_{L_i}$ is equal to $\mathrm{ad}_{L_i}\, x_{i,s}$, hence this is a semisimple endomorphism on $L_i$. Note that $[L_iL_j] \subset L_i \cap L_j = 0$ for $i \neq j$. Hence, $\mathrm{ad}_L\, x_{i,s}|_{L_j} = 0$ for $i \neq j$. It follows that $\mathrm{ad}_L\, x_{i,s}$ is a semisimple endomorphism on $L$. Let $u := \sum_{i=1}^t x_{i,s}$. Then $\mathrm{ad}_L\, u$ is is a semisimple endomorphism on $L$. Similarly, denoting by $v = \sum_{i=1}^t x_{i,n}$ the sum of the nilpotent parts, it follows that $\mathrm{ad}_L\, v$ is a nilpotent endomorphism on $L$. Clearly, $x = u + v$. Moreover,

$$[uv] = \left[\sum_{i=1}^t x_{i,s} \sum_{i=1}^t x_{i,n}\right] = \sum_{i,j} [x_{i,s}x_{j,n}] = \sum_{i=1}^t [x_{i,s}x_{i,n}] = 0 \; .$$

Hence, $u = x_s$ and $v = x_n$ by uniqueness of the Jordan decomposition. ■

So, let's decompose $H$. Let $H_i := L_i \cap H$. It follows from Lemma 14.2 that $H_i$ consists of semisimple elements of $L_i$, hence this is a toral subalgebra of $L_i$. In fact, it is a *maximal* toral subalgebra: any toral subalgebra in $L_i$ larger than $H_i$ would also be toral in $L$, centralize all $H_j$ with $j \neq i$, and generate with them a toral subalgebra of $L$ larger than $H$. We claim that $H = \bigoplus_{i=1}^t H_i$.[1] Clearly, $H_i \cap H_j = 0$ if $i \neq j$. To see that $H$ is the sum of the $H_i$, let $x \in H$. Then $x = \sum_{i=1}^t x_i$ with $x_i \in L_i$. Since $H$ is abelian by Lemma 8.1, we have $[HH] = 0$, so in particular, $[xH] = 0$, hence $[xH_i] = 0$. For $j \neq i$ we have $[x_jL_i] = 0$, so $[x_jH_i] = 0$. We thus must have $[x_iH_i] = 0$, i.e. $x_i \in C_{L_i}(H_i)$. But $H_i$ is a maximal toral subalgebra in $L_i$, thus self-centralizing by Proposition 8.4, i.e. $x_i \in H_i$. This shows that $H$ is the sum of the $H_i$.

Now, let $\Phi_i$ be the root system of $L_i$ with respect to $H_i$. Let $E_i$ be the vector space $\Phi_i$ lives in, i.e. $E_i = \mathbb{R}\Phi_i \subset H_i^*$. If $\alpha \in \Phi_i$, we can just as well view $\alpha$ as a linear function on $H$ by setting $\alpha(H_j) = 0$ for $j \neq i$. Then $\alpha$ is clearly a root of $L$ relative to $H$, with $L_\alpha \subset L_i$. Conversely, if $\alpha \in \Phi$, then $[H_iL_\alpha] \neq 0$ for some $i$ (otherwise, $L_\alpha$ would lie in the centralizer of $H$ in $L$, but $H$ is self-centralizing by Proposition 8.4), and we must have $L_\alpha \subset L_i$, so $\alpha|_{H_i}$ is a root of $L_i$ relative to $H_i$. This shows that we can view $\Phi_i$ naturally as a subset of $\Phi$ and we have a

[1] This is somehow not really explained in Humphreys. I don't think it's obvious.

decomposition $\Phi = \Phi_1 \cup \ldots \Phi_t$. Moreover, the $L_i$ are mutually orthogonal with respect to the Killing form. Each $\Phi_i$ is the root system of a simple Lie algebra $L_i$, hence it is irreducible by Proposition 14.1, therefore the $\Phi_i$ are the irreducible components of $\Phi$.

Proven in Section 5

Hence, when we want to understand to which extend a semisimple Lie algebra is characterized by its root system, it is enough to consider *simple* Lie algebras and their (irreducible) root systems.

**14.2. Isomorphism theorem.** To define a morphism of Lie algebras, it is helpful to have a special generating system of the Lie algebra at hand on which we can explicitly define the morphism.

***Proposition 14.3.*** *Let $L$ be a semisimple Lie algebra, let $H$ a maximal toral subalgebra, and let $\Delta$ be a base of the corresponding root system $\Phi$. Then $L$ is generated (as a Lie algebra) by the root spaces $L_{\pm\alpha}$ for $\alpha \in \Delta$.*

*Proof.* Let $\beta$ be an arbitrary positive root (relative to $\Delta$). By Corollary 10.7 we can write $\beta$ as $\beta = \alpha_1 + \ldots + \alpha_s$ with $\alpha_i \in \Delta$ and such that each partial sum $\alpha_1 + \ldots + \alpha_i$ is a root. From Lemma 8.3 we know that $[L_\gamma L_\delta] \subseteq L_{\gamma_\delta}$. By induction on $s$, we then easily see that $L_\beta$ lies in the subalgebra of $L$ generated by $L_\alpha$ for $\alpha \in \Delta$. Similarly, if $\beta$ is negative, then $L_\beta$ lies in the subalgebra of $L$ generated by all $L_{-\alpha}$ for $\alpha \in \Delta$. Since $L = H \oplus \bigoplus_{\alpha \in \Delta} L_\alpha$ and $H = \sum_{\alpha \in \Phi} [L_\alpha L_{-\alpha}]$ by Proposition 8.5, the proposition follows. ■

By Proposition 8.5, the root spaces $L_\alpha$ are one-dimensional. Hence, if we choose an $\mathfrak{sl}_2$-triple $(x_\alpha, h_\alpha, y_\alpha)$ for each $\alpha \in \Delta$, then the set $\{x_\alpha, y_\alpha \mid \alpha \in \Delta\}$ is a generating set of $L$. We call this a **standard set of generators**. Now, we come to the promised theorem.[2]

***Theorem 14.4.*** *If the root systems $\Phi$ and $\Phi'$ of semisimple Lie algebras $L$ and $L'$ are isomorphic, then $L$ and $L'$ are isomorphic.*

*Proof.* Let $(E, \Phi)$ and $(E', \Phi')$ be the root systems of $L$ and $L'$ for a choice of maximal toral subalgebras $H$ and $H'$. By assumption, there is a vector space isomorphism $f \colon E \to E'$ with $f(\Phi) = \Phi'$ and preserving the Cartan integers. Let $\Delta \subset \Phi$ be a base. Then $\Delta' \coloneqq f(\Delta)$ is a base of $\Phi'$. Choose $\mathfrak{sl}_2$-triples $(x_\alpha, h_\alpha, y_\alpha)$ in $L$ for all $\alpha \in \Delta$. Likewise, choose $\mathfrak{sl}_2$-triples $(x'_{\alpha'}, h'_{\alpha'}, y'_{\alpha'})$ in $L'$ for all $\alpha' \in \Delta'$, where we set $\alpha' \coloneqq f(\alpha)$.

We first assume that $L$ and $L'$ are simple; the general case will then follow easily. Let $K$ be the subalgebra of $L \oplus L'$ generated by the elements

$$\hat{x}_\alpha \coloneqq x_\alpha \oplus x'_{\alpha'} \in L_\alpha \oplus L'_{\alpha'} \quad \text{and} \quad \hat{y}_\alpha \coloneqq y_\alpha \oplus y'_{\alpha'} \in L_{-\alpha} \oplus L'_{-\alpha'} .$$

[2] I really don't like how Humphreys [1, Section 14.2] is proceeding with the isomorphism theorem, especially with the sentences "However, the root system axioms are unaffected if we multiply the inner product by a positive real number. Therefore, it does not harm to assume that the isomorphism comes from an isometry.", which doesn't make sense to me. I have reorganized things a bit to make this more precise. Ugh.

We will first show that $K \neq L \oplus L'$. Since we assumed that $L$ and $L'$ are simple, their root systems $\Phi$ and $\Phi'$ are irreducible by Proposition 14.1. Hence, by Lemma 10.16 there is a unique maximal root $\beta \in \Phi$ and $\beta' \in \Phi'$. The map $f$ preseves the partial ordering on roots since if $\gamma \in \Phi$ is positive, then $\gamma = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with non-negative coefficients $k_\alpha$ and so $f(\gamma)$ is likewise a non-negative linear combination of the simple roots $\alpha' \in \Delta'$, hence positive. It follows that $f$ must map $\beta$ to $\beta'$. Now, let $0 \neq x \in L_\beta$ and $0 \neq x' \in L'_{\beta'}$. Set $\hat{x} := x \oplus x'$. Let $V$ be the subspace of $L \oplus L'$ spanned by all $[\hat{z}\hat{x}]$ for $\hat{z} \in K$. Note that for any $\alpha \in \Delta$ we have $[x_\alpha x] \in L_{\alpha+\beta} = 0$ by maximality of $\beta$. Similarly, $[x'_{\alpha'} x'] = 0$. So, $[\hat{x}_\alpha \hat{x}] = 0$ for all $\alpha \in \Delta$. Since $K$ is generated by the $\hat{x}_\alpha$ and $\hat{y}_\alpha$, it follows that $V$ is already spanned by the elements

$$[\hat{y}_{\alpha_1}[\hat{y}_{\alpha_2} \ldots [\hat{y}_{\alpha_m}\hat{x}]]] \in L_{\beta - \sum_i \alpha_i} \oplus L'_{\beta' - \sum_i \alpha'_i}$$

for $\alpha_i \in \Delta$, not necessarily distinct. Hence, with these brackets we always move down in the weights. Since $\beta, \beta'$ are maximal, it follows that the intersection $V \cap (L_\beta \oplus L_{\beta'})$ is one-dimensional, spanned by $\hat{x}$. But $L_\beta \oplus L_{\beta'}$ is two-dimensional (direct sum of two one-dimensional spaces), so we must have $V \neq L \oplus L'$. We wanted to prove that $K \neq L \oplus L'$. If we had $K = L \oplus L'$, then $V$ would be an ideal in $L \oplus L'$ by definition of $V$, non-trivial by what we just deduced, so either $V = L$ or $V = L'$ since both $L, L'$ are simple. But this cannot be true since $\hat{x} \in V$ but neither $\hat{x} \in L$ not $\hat{x} \in L'$.

Now, let $\pi\colon K \to L$ and $\pi'\colon K \to L'$ be the projections, restricted to $K$. Clearly, these are morphisms of Lie algebras. The map $\pi$ is surjective, since in the image we get all the vectors $x_\alpha, y_\alpha$ for $\alpha \in \Delta$ and these generate $L$ by Proposition 14.3. Similarly, $\pi'$ is surjective. We claim that these maps are also injective. Suppose $\pi'$ were not injective. Then there is an element $\hat{z} = z \oplus 0 \in K$ with $z \neq 0$. The ideal $I$ in $K$ generated by $z$ is obtained by repeatedly taking brackets with the generators. Nothing happens in the second component as this is zero, so $I$ looks like $J \oplus 0$. The $J$ is stable under all $x_\alpha, y_\alpha$, hence it is an ideal in $L$ since $L$ is generated by the $x_\alpha, y_\alpha$ by Proposition 14.3. But $L$ is simple, so $J = L$, i.e. $I = L \oplus 0$. This shows that $L \oplus 0 \subset K$. Similarly, we see that $0 \oplus L' \subset K$, hence $L \oplus L' \subset K$, a contradiction to what we have proven above. Hence, $\pi'$, and similarly $\pi$, is an isomorphism. We thus have isomorphisms

$$L \xleftarrow{\;\pi\;} K \xrightarrow{\;\pi'\;} L'$$

from which we get an isomorphism $\phi\colon L \to L'$. Note that this isomorphism maps $x_\alpha$ to $x'_{\alpha'}$ and $y_\alpha$ to $y'_{\alpha'}$ for all $\alpha \in \Delta$. Moreover, since $[x_\alpha y_\alpha] = h_\alpha$ and $\phi$ is a morphism of Lie algebras, we also have

$$\phi(h_\alpha) = [\phi(x_\alpha)\phi(y_\alpha)] = [x'_{\alpha'} y'_{\alpha'}] = h'_{\alpha'} \ .$$

As the last step, consider the general case when $L$, $L'$ are note necessarily simple. Let $L = L_1 \oplus \ldots \oplus L_t$ be the decomposition of $L$ into simple ideals. We have discussed in Section 14.1 that the root system $\Phi_i$ of $L_i$ with respect to the maximal toral subalgebra $H_i = L_i \cap H$ are precisely the irreducible components of $\Phi$.

If we have an isomorphism $f\colon (E,\Phi)\to(E',\Phi')$ of root systems, then the $f(\Phi_i)$ are the irreducible components of $\Phi'$. Again by the discussion in Section 14.1, $\Phi_i'$ is the root system of a simple ideal $L_i'$ of $L_i$ with respect to $H_i' = L_i' \cap H'$. Now, $\Phi_i$ and $\Phi_i'$ are isomorphic. Hence, $L_i$ and $L_i'$ are isomorphic by the simple case we have already proven. Taking the direct sum of these isomorphisms yields an isomorphism $L \to L'$. ∎

In the theorem we have given an explicit construction of an isomorphism $\phi\colon L \to L'$ of Lie algebras starting from an isomorphism $f\colon (E,\Phi)\to(E',\Phi')$ of their root systems: it maps chosen $\mathfrak{sl}_2$-triples on a base $\Delta$ to chosen $\mathfrak{sl}_2$-triples on the image base $f(\Delta)$. Since $\Phi$ spans $H^*$, we get from $f$ a vector space isomorphism $f\colon H^* \to (H')^*$. The inner product induces an isomorphism $H^* \to H$ mapping $\alpha$ to the element $t_\alpha \in H$ characterized by $\alpha = (t_\alpha, -)$. Similarly, we have an isomorphism $(H')^* \to H'$. Hence, from $f$ we get a vector space isomorphism $f^\flat\colon H \to H'$. How does this map relate to the restriction of the Lie algebra morphism $\phi$ to $H$? If both maps were the same, we can rightly say that $\phi$ is an **extension** of $f$ to a Lie algebra isomorphism. By construction of $\phi$, it maps $h_\alpha = 2t_\alpha/(\alpha,\alpha)$ to $h'_{\alpha'} = 2t'_{\alpha'}/(\alpha',\alpha')$. By construction of $f^\flat$, it maps $t_\alpha$ to $t'_{\alpha'}$. Hence, $f^\flat(h_\alpha) = 2t'_{\alpha'}/(\alpha,\alpha)$. We thus have

$$f^\flat(h_\alpha) = h'_{\alpha'} = \phi(h_\alpha) \Leftrightarrow (\alpha',\alpha') = (\alpha,\alpha)\ . \tag{14.1}$$

So, only if $f$ is an *isometry*, the induced Lie algebra isomorphism $\phi$ is actually an extension of $f$! In general, $f^\flat$ will scale the vectors $h_\alpha$ by a common positive real number for each irreducible component of the root system thanks to the following lemma.[3]

***Lemma 14.5.***

*(a) An isomorphism between two irreducible root systems is* ***conformal****, i.e. it scales the inner product by a constant positive real number.*

*(b) An automorphism of an irreducible root system is always an isometry.*

*Proof.* We first prove the following: for any pair of roots $\alpha,\beta$ in an irreducible root system $\Phi$ there is a sequence of roots $\alpha = \alpha_1,\dots,\alpha_n = \beta$ which are consecutively non-orthogonal, i.e. $(\alpha_i,\alpha_{i+1}) \neq 0$ for all $i$. Write $\alpha \sim \beta$ if there is such a sequence. This is clearly an equivalence relation on $\Phi$. Let $\Phi_\alpha$ be the equivalence class of $\alpha$ and let $\Phi'_\alpha := \Phi \setminus \Phi_\alpha$. These two subsets are orthogonal since if some $\gamma' \in \Phi'_\alpha$ were not orthogonal to some $\gamma \in \Phi_\alpha$, then $\alpha \sim \gamma$ and $\gamma \sim \gamma'$, so $\alpha \sim \gamma'$, a contradiction. So, $\Phi = \Phi_\alpha \cup \Phi'_\alpha$ is an orthogonal decomposition. Since $\Phi$ is irreducible and $\Phi_\alpha$ is non-empty, we must have $\Phi'_\alpha = \emptyset$, proving the claim.

Now, let $f\colon (E,\Phi)\to(E',\Phi')$ be an isomorphism of irreducible root systems. Let $\alpha,\beta \in \Phi$. By definition, $f$ preserves the Cartan integers, so

$$\frac{2(f(\alpha),f(\beta))}{(f(\alpha),f(\alpha))} = \langle f(\alpha), f(\beta)\rangle = \langle \alpha,\beta\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\ .$$

[3] I've learned about this lemma from https://mathoverflow.net/questions/60935/a-question-on-the-root-systems-of-simple-lie-algebras-in-the-90-degree-case.

Hence,

$$(f(\alpha), f(\alpha)) = \frac{(f(\alpha), f(\alpha))}{(\alpha, \alpha)}(\alpha, \beta) = c_\alpha(\alpha, \beta)\ , \tag{14.2}$$

where

$$c_\alpha := \frac{(f(\alpha), f(\alpha))}{(\alpha, \alpha)} \in \mathbb{R}_{>0}$$

is the scaling factor for $\alpha$. Exchanging $\alpha$ and $\beta$ above gives

$$(f(\beta), f(\alpha)) = c_\beta(\beta, \alpha)\ .$$

Since the inner product is symmetric, we get

$$(f(\alpha), f(\beta)) = c_\beta(\alpha, \beta)\ . \tag{14.3}$$

Combining (14.2) and (14.3), we see that $c_\alpha = c_\beta$ whenever $(\alpha, \beta) \neq 0$. Due to the claim we have proven at the beginning, we can connect any two roots by a sequence of non-orthogonal roots, hence the scaling factor $c_\alpha$ is the same factor $c$ for every root $\alpha$. The equations above show that the inner product between all roots is scaled by $c$, and since the roots span $E$, this proves the claim.

Finally, let $f$ be an automorphism of an irreducible root system $\Phi$. By the above, $f$ is conformal, i.e. it scales lengths by a common positive real number $\sqrt{c}$. Let $l$ be the norm of the longest root in $\Phi$. Then $\sqrt{c}l$ is the norm of the longest root in $f(\Phi)$. Since $f(\Phi) = \Phi$, we have $\sqrt{c}l = l$, so $c = 1$ and $f$ is an isometry. ■

**14.3. Automorphisms.** As a special case of Theorem 14.4 we obtain that every automorphism $f$ of the root system $\Phi$ of a semisimple Lie algebra $L$ induces a Lie algebra automorphism $\phi$ of $L$. If the automorphism happens to be an isometry, then $\phi$ is really an extension of $f$ to $L$. By Lemma 14.5 this holds automatically if $L$ is simple.

## References

[1] J. E. Humphreys. *Introduction to Lie algebras and representation theory.* Vol. 9. Graduate Texts in Mathematics. Third printing, revised. Springer-Verlag, New York-Berlin, 1980.

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